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A COMPARISON OF GRADIENT ESTIMATION TECHNIQUES FOR EUROPEAN CALL OPTIONS

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ABSTRACT

Assuming the underlying assets follow a Variance-Gamma (VG) process, we consider the problem of estimating gradients of a European call option by Monte Carlo simulation methods. In this paper, we compare indirect methods (finite difference techniques such as forward differences) and two direct methods: infinitesimal perturbation analysis (IPA) and likelihood ratio (LR) method. We conduct simulation experiments to evaluate the efficiency of different estimators and discuss the advantage and disadvantage of each method.

JEL: G13, G15, G17

KEYWORDS: Greeks, IPA, LR, Variance-Gamma

INTRODUCTION

Gradients estimates have been useful in hedging risks in markets in the finance community. Thus many techniques of calculation of gradients including direct methods and indirect methods have been broadly developed. Many studies have been obtained to study the gradients of estimation under a geometric Brownian motion (GBM) model. However, the GBM model has some imperfections in illustrating the statistical properties of empirical results of market prices. In this paper, we assume stock prices follow a Variance-Gamma (VG) process and develop gradient estimates of a European call option under this assumption. The Variance Gamma process is one of the Levy processes, which are determined by a random time change. It is a pure-jump process with finite moments and no diffusion component. The VG process has been studied in a vast literature and empirical evidence shows that it can yield much better fits to stock prices than the geometric Brownian motion process.

In this paper, we first price a European call option and then turn to gradient estimation to calculate the Greeks by indirect method: forward difference (FD), the direct methods of IPA and LR. Finally, an analysis of the strengths and weakness of each method is provided. The remaining of this paper is organized as follows. A literature review of gradient estimation techniques and Variance Gamma processes are first provided. Then, the introduction of Greeks which is also called the sensitivities of options is shown. In the third part, details of VG processes, as well as gradient estimation techniques including forward direct method (FD), IPA and LR are provided. Furthermore, gradient estimators of Greeks of options under the VG model are shown. Finally, a numerical experiment of estimating Greeks of a European call option is conducted using estimators we calculated. In the last section, analysis of results from the numerical experiment is provided.

LITERATURE REVIEW

A Variance Gamma (VG) Process was introduced to the finance community as a model for log-price returns and option pricing by Madan and Seneta (1990). Madan and Milne (1991) consider equilibrium option pricing for a symmetric variance gamma process in a representative agent model; while Madan, Carr and Chang (1998) develop the method of pricing options by a Variance Gamma process. Fu (2007) gives a general introduction to the VG process in the context of stochastic (Monte Carlo) simulations and shows how to price and simulate stock prices. Cao and Fu (2010) estimate Greeks of Mountain Range options with respect to a Variance Gamma process.

The Greeks in the definition of finance community are the quantities representing the sensitivities of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. Each Greek letter measures the sensitivity of option prices to stock prices, thus it has been broadly applied in hedging risks. For example, "Delta" is necessary for delta hedging, see Cao and Guo (2011-3) employ deltas under a Black-Scholes model to estimate hedging profits. Cao and Guo (2001-2) analyze hedging profits from delta hedging under a VG model and a Black-Scholes model; while Cao and Guo (2011-4) compare results from delta hedging of a European call option w.r.t. a VG process and a geometric Brownian motion (GBM), respectively, using deltas by IPA method. The Greeks can also be employed to conduct other hedging strategies such as Gamma hedging etc., of which are introduced in Hull (2003).

Gradient estimation technique is a widely used technique to calculate the Greeks. It was first applied to option pricing using infinitesimal perturbation analysis (IPA) for both European and American options by Fu and Hu (1995). Then, both IPA and LR methods are applied in Broadie and Glasserman (1996) to European and Asian option with respect to (w.r.t.) Geometric Brownian motion; see also Glasserman (2004) reviews various Monte Carlo Methods for financial engineering. Fu (2007) reviews various methods of gradient estimation in stochastic simulation, including both direct and indirect methods; see also Fu (2008) reviews techniques and applications to derivative securities. Cao and Guo (2011-1) employ the estimation techniques to estimate gradients of a European call option following a VG model.

VARIANCE-GAMMA PROCESS

The Variance Gamma Process is a Levy process, which is of independent and stationary increments. There are two ways to define a VG process:

First, a VG process can be defined as Gamma-time-changed Brownian motion with the subordinator being a gamma process, say GVG. Let $W_t$denote the standard Brownian motion, $B_t^{(\mu,\sigma)} = \mu t + \sigma W_t$ denote the Brownian motion with constant drift rate μ and volatilityσ, $\gamma_t^{(v)}$ be the gamma process with drift $\mu = 1$ and variance parameter v. The representation of the VG process is:

$$X_t = B_{\gamma_t^{(v)}}^{(\theta,\sigma)} = \theta\gamma_t^{(v)} + \sigma W_{\gamma_t^{(v)}}. \tag{1}$$

Second, the VG process is the difference of two gamma processes, say DVG. Let $\gamma_t^{(\mu,v)}$be the gamma process with drift parameter μ and variance parameter v, the representation of the VG process as difference of gamma process is:

$$X_t = \gamma_t^{(\mu_+,v_+)} - \gamma_t^{(\mu_-,v_-)}, \tag{2}$$

where $\mu_\pm = (\sqrt{\theta^2 + \frac{2\sigma^2}{v}} \pm \theta)/2,$ and $v_\pm = \mu_\pm^2 v.$

Under the risk-neutral measure, with no dividends and constant risk-free interest rate r, the stock price is given by $S_t = S_0 \exp\big((\gamma + w)t + X_t\big),$ where $\omega = \ln\left(1 - \theta v - \frac{\sigma^2 v}{2}\right)/v$ is the parameter that makes the discounted asset price a martingale.

The density function of the log-price $Z = \ln(\frac{S_t}{S_0})$ as proposed by Madan and Seneta (1990) is:

$$h(z) = \frac{2\exp\left(\frac{\theta x}{\sigma^2}\right)}{v^{\frac{t}{v}}\sqrt{2\pi}\sigma\Gamma\left(\frac{t}{v}\right)}\left(\frac{x^2}{\frac{2\sigma^2}{v}+\theta^2}\right)^{\frac{t}{2v}-\frac{1}{4}} K_{\left(\frac{t}{v}-\frac{1}{2}\right)}\left(\frac{1}{\sigma^2}\sqrt{x^2\left(2\frac{\sigma^2}{v}+\theta^2\right)}\right) \tag{3}$$

where K is the modified Bessel function of 2nd kind, and $x = z - rt - \frac{t}{v}\ln\left(1 - \theta v - \frac{\sigma^2 v}{2}\right)$.

GREEKS

Greeks are quantities representing sensitivities of derivatives, such as options, see Hull (2003). Each Greek letter measures a different dimension to the risk in an option position and the aim of a trader is to manage the Greeks so that all risks are acceptable. In this paper, we study the Greeks such as Delta, Rho and Theta defined in the following:

Delta: Δ is defined as the rate of change of the option price w.r.t. the underlying asset price. It is the slope of the curve that relates the option price to the underlying asset price. In general, $\Delta = \frac{\partial V}{\partial S}$.

Vega: v is the rate of change of the value of the portfolio of option w.r.t. the volatility of the underlying asset price. It measures the sensitivity of the value of a portfolio to the volatility, i.e.,

$$v = \frac{\partial V}{\partial \sigma} .$$

Rho: ρ is the rate of change of the value of the portfolio of option w.r.t. the interest rate. It measures the sensitivity of the value of a portfolio to interest rates. It is defined as: $\frac{\partial V}{\partial r}$.

Theta: θ is the rate of change of the value of the portfolio of option w.r.t. the passage of time with all else remaining the same. It measures the sensitivity of the value to the passage of time. It is defined as: $\theta = \frac{\partial V}{\partial t}$

GRADIENT ESTIMATION TECHNIQUE

In this paper, we focus on calculating gradient estimates of the price of a European call option depending on various parameters of a VG model. We then calculate the derivatives of the price with respect to these parameters separately.

We begin with $J(\xi)$, the objective function which depends on the parameterξ, and calculate $\frac{dJ(\xi)}{d\xi}$.

Suppose the objective function is an expectation of the sample performance measure L, that is:

$$J(\xi) = E[L(\xi)] = E[L(X;\xi)] \tag{4}$$

Where X is dependent on ξ. By the law of the unconscious statistician, the expectation can be written as:

$$E[L(X)] = \int y dF_L(y) = \int L(x) d\, F_X(x), \tag{5}$$

where F_L is the distribution of L and F_X is the distribution of input random variables X.

Indirect Methods

The indirect method of estimating a gradient at ξ is simply to use finite difference, i.e., perturbing the value of each component of ξ separately while holding all the other components still.

The one-sided forward difference gradient estimator in the i-th direction is: $\frac{\hat{J}(\xi+c_i e_i)-\hat{J}(\xi)}{c_i}$, where c_i is the scalar perturbation in the i-th direction and e_i is the unit vector in the i-th direction.

Direct Methods (IPA and LR)

IPA estimates require the integrability condition which is easily satisfied when the performance function is continuous with respect to the given parameter. Assume we can interchange the expectation and differentiation, the IPA estimate is:

$$\frac{dE[L(X)]}{d\xi} = E\left[\frac{dL(X)}{d\xi}\right] = \int_0^1 \frac{dL}{dX}\frac{dX(\xi)}{d\xi}du, \tag{6}$$

and the estimator is:

$$\frac{dL}{dX}\frac{dX(\xi)}{d\xi} \tag{7}$$

From the Lebesgue dominated convergence theorem, the condition of uniform integrability of $\frac{dL}{dX}\frac{dX(\xi)}{d\xi}$ must be satisfied to make the interchangeability.

For LR, the probability density function f of X is differentiable. The Likelihood Ratio method is:

$$\frac{dE[L(X)]}{d\xi} = \int_{-\infty}^{+\infty} L(x)\frac{df(x,\xi)}{d\xi}dx = \int_{-\infty}^{+\infty} L(x)\frac{dlnf(x,\xi)}{d\xi}f(x)dx$$

and the estimator is

$L(x)\frac{dlnf(x,\xi)}{d\xi}f(x)$, where $\frac{dlnf(x,\xi)}{d\xi}$ is the score function. From the Lebesgue dominated convergence theorem, the condition of uniform integrability of $L(x)\frac{dlnf(x,\xi)}{d\xi}f(x)$ must be satisfied to make the interchangeability. We employ indirect methods (FD) and direct methods (IPA and LR) to calculate the gradient estimation in the following paper.

GRADIENTS OF A EUROPEAN CALL OPTION

Call option gives the buyer the right, not the obligation to buy certain amount of financial instrument from the seller at a certain time for a certain price. The payoff function of the European call option with expiring time T, strike price K and risk free interest rate r is: $V_T = e^{-rT}(S_T - K)^+$, where $S_T = S_0\exp((r+\omega)T + X_T)$, and X_T follows the VG process. We have 2 different ways to represent the VG process X_T as in Equation (2) and Equation (3). We estimate the Greeks in these two ways.

IPA for a European Call Option

The gradient w.r.t T does not satisfy the condition of interchangeability, which means IPA method can not be applied to this gradient. The estimators for other gradients of a European call option for IPA method are as follows:

Table 1: IPA Estimators for European Call options

Greek	IPA Estimators
Delta	$\frac{dV_T}{dS_0} = e^{-rT} I_{\{S_T>k\}} \frac{dS_T}{dS_0}$
Rho	$\frac{dV_T}{dr} = -Te^{-rT}(S_T - K)^+ + e^{-rT} I_{\{S_T>k\}} \frac{dS_T}{dr}.$
Vega	$\frac{dV_T}{d\sigma} = e^{-rT} I_{\{S_T>k\}} \frac{dS_T}{d\sigma}.$
Theta	$\frac{dV_T}{dT} = -re^{-rT}(S_T - K)^+ + e^{-rT} I_{\{S_T>k\}} \frac{dS_T}{dT}$
Gradient w.r.t. θ	$\frac{dV_T}{d\theta} = e^{-rT} I_{\{S_T>k\}} \frac{dS_T}{d\theta}$

Notes: This table shows the IPA estimates for European call options by assuming the stock price follows a Variance-Gamma process. The Variance-Gamma process could be a GVG or DVG process. Delta denotes the gradient with respect to the spot price S_0. Rho denotes the gradient with respect to the risk-free interest rate r. Vega denotes the gradient with respect to σ. Theta denotes the gradient with respect to the maturity time T.

LR for a European Call Option

Since the density doesn't contain S_0 or r, we could not use LR to estimate vega and rho. The gradient w.r.t. θ does not satisfy the condition of interchangeability, which means LR method can not be applied to this gradient. The other gradients could be calculated as follows:

Table 2: LR Estimators for European Call options

Greek	LR Estimators
Gradient w.r.t. σ	$\frac{d\,E[V_T]}{d\sigma} = \int_0^\infty e^{-rT}(S_0 e^Z - K)^+ \frac{dlnh(z)}{d\sigma} h(z)dz$
Theta (Gradient w.r.t. T)	$\frac{d\,E[V_T]}{dT} = \int_0^\infty e^{-rT}(S_0 e^Z - K)^+ (-r + \frac{dlnh(z)}{dT})h(z)dz$

Notes: This table shows the LR estimates for European call options by assuming the stock price follows a Variance-Gamma process. The Variance-Gamma process could be a GVG or DVG process. T denotes the maturity time.

Numerical Experiment

Using the formulas of estimators above, we apply Monte Carlo to do the estimation from 10000 sample paths. With $K = 10, r - \delta = 0.057$, $v = 0.2686$, $\theta = 0.1436, \sigma = 0.1213$ and $T = 0.2$. We get the numerical results in the table below:

Table 3: Numeral Results of Gradients

GVG	Delta	Rho	Vega	$\frac{dV_T}{dT}$	$\frac{dV_T}{d\theta}$
FD	0.4746	0.9010	1.3703	0.6443	0.4713
StdErr	0.0052	0.0097	0.0349	0.6559	0.0212
IPA	0.4707	0.8862	1.3578		0.4621
StdErr	0.0052	0.0098	0.0349		0.0211
LR			1.7603	0.7012	
StdErr			0.2036	0.0236	
DVG					
FD	0.4794	0.9100	1.3883	0.5942	0.5022
StdErr	0.0052	0.0097	0.0320	0.7987	0.0171
IPA	0.4746	0.8925	1.1200		0.5512
StdErr	0.0052	0.0098	0.0013		0.0120
LR			1.3651	0.6792	
StdErr			0.1688	0.0317	

Notes: This table shows the mean values and standard errors of results of all Greeks for European call options by assuming the stock price follows a Variance-Gamma process. The Variance-Gamma process could be a GVG or DVG process. GVG is the Gamma-time-changed Brownian motion. DVG is the difference of two Gamma processes. We apply the results of estimating Greeks by three methods FD, IPA and LR. FD denotes the forward difference method, LR denotes the LR method. StdErr denotes the standard error of the simulation results. Delta denotes the gradient with respect to the spot price S_0. *Rho denotes the gradient with respect to the risk-free interest rate* r. *Vega denotes the gradient with respect to* σ. *Panel A shows the results under a GVG process; Panel B shows the results under a DVG process.*

CONCLUSIONS

Assuming stock prices follow a Variance-Gamma process, we employ several methods in the gradient estimation techniques to estimate the Greeks of a European call option. The gradient estimators of gradients through three methods including the finite difference, the infinitesimal perturbation analysis (IPA) method and the likelihood ratio (LR) method are provided in Table 1 and Table 2. A numerical experiment is conducted. From the results in the table 3, we could draw the following conclusions. FD method is closest to the true value of gradients, but may get large standard error. Moreover, compare to IPA and LR methods, FD method is more time-consuming, since it requires running the simulation on one sample path twice. But estimators of IPA and LR of some gradients do not satisfy the condition of changing integral, i.e., the Lebesgue dominated theorem. Therefore, these two methods have some limitations in calculating some of the gradients. Furthermore, from the results in table 3, we could conclude that IPA method is more accurate than LR method.

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