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AN EXPOSITION ON THE MATHEMATICS AND ECONOMICS OF OPTION PRICING

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ABSTRACT

The application of options pricing theory to value irreversible investment decisions has witnessed a marked increase over the last decade. For instructional and simplified applications, the Black-Scholes model is commonly demonstrated due to its tractability and acceptance in the finance community. This paper provides a detailed mathematical exposition of the Black-Scholes model. The main contribution of this paper is the step-by-step instructional account of the Black-Scholes model that can be used directly in the classroom to introduce stochastic calculus, arbitrage-free valuation, and option-pricing theory. In contrast with most Black-Scholes derivations found in the pedagogical literature, this paper develops the fair option price from an economic equilibrium perspective. Through this approach, it is hoped the reader will comprehend both the mathematics and economics underlying option pricing theory, as both are equally important.

JEL: A22, A23, C02, G00, M19

KEYWORDS: Options Pricing, Black-Scholes Model, Stochastic Calculus, Pedagogy

INTRODUCTION

Black and Scholes (1973) and Merton (1973) published their seminal work on pricing financial options using a continuous-time model that is known as the Black-Scholes equation. Their work led to an explosion of financial risk management practices within firms, and some attribute their efforts to the successful Chicago Board Options Exchange (CBOE) which today has an estimated nominal value of $100 trillion.

In order to fully comprehend and appreciate options literature, a basic understanding of financial option pricing mathematics is required. As the majority of these mathematical efforts have been initiated and developed under the financial economist umbrella, many business professionals and non-finance experts lack the appropriate exposure to the material. Numerous texts exist on the subject; however, a detailed step-by-step and line-by-line treatment of the Black-Scholes equation are not easily found in texts and related materials.

As such, the purpose of this paper is to formally develop the mathematics and economics behind the Black-Scholes equation. In contrast with most Black-Scholes derivations found in the pedagogical literature, this paper develops the fair option price from an economic equilibrium perspective. Through this approach, it is hoped the reader will comprehend both the mathematics and economics underlying option pricing theory, as both are equally important. This paper does not develop new theories, but instead contributes with its detailed exposition of existing material. For those interested in options pricing, we believe this detailed presentation provides a clear and concise path to comprehend and/or teach basic option pricing mathematics. We have used the techniques in this paper for our *Financial Engineering* courses and find it appropriate for advanced undergraduate and graduate students across numerous disciplines.

This paper is organized as follows. After a brief literature review, Section 2 discusses the mathematics necessary for deriving the Black-Scholes model and its relevance for asset pricing in general. Section 3 develops the Black-Scholes partial differential equation using appropriate mathematical and economic principles. Section 4 provides concluding remarks.

LITERATURE REVIEW

In order to understand option pricing models, analysts must first understand the tools used to develop these models. This module is a brief introduction to stochastic calculus and its import in asset pricing. Far from being a mere theoretical development, stochastic calculus is a practical method that can be used by both academics and market professionals. An introduction to stochastic calculus can be found in Baxter and Rennie (1996), Hoel et. al (1972), Kushner (1995), Merton (1990), Neftci (2000), and Wilmot et. al (1995). Dixit and Pindyck (1994) and Copeland and Antikarov (2001) demonstrate these financial option-pricing techniques applied to capital budgeting decision-making.

Understanding the derivation of the Black-Scholes-Merton equation will assist scholars to understand numerous other closed-form equations to value options. The option to exchange one asset for another was developed by Margrabe (1978). Fischer (1978) developed an equation to for option valuation with an uncertain exercise price. Geske (1979) developed an equation to value compound options with deterministic exercise prices. Carr (1988) developed a compound option equation with stochastic exercise prices.

Finally, the techniques demonstrated in this paper allow a more intuitive interpretation to understand numerous lattice approaches. Cox, Ross, and Rubinstein (1979) developed a binomial discrete-time option valuation technique. Boyle (1986) developed a trinomial tree, and Boyle (1988) developed a five-jump tree. Madan, Milne, and Shefrin (1989) generalized the binomial model to the multinomial case.

<u>Introduction to Stochastic Calculus</u>

For any option pricing model the objective is to find an analytical function, $C\big(S(t),t\big)$, that expresses the value of an option in terms of the underlying asset price, $S(t)$, and time, t. A fundamental step to obtain this pricing equation begins with obtaining the dynamics of the option. The reason is simple. Understanding the dynamics of the option allows an analyst to either internalize the contribution the option contributes to a portfolio or to formulate a reasonable forecast of the asset's price at some future point in time.

Deterministic Calculus: To illustrate the usefulness of the options dynamics, consider a simple example of a derivative asset, whose value is a function of a stock price and time, namely $C\big(S(t),t\big)$. Furthermore, presume the dynamics of the stock price, $S(t)$, are deterministic or non-random. In this particular case, if the option is a continuously differentiable function with respect to the stock price and time, then from ordinary calculus a Taylor series expansion may be used to express the dynamics of the option. Using a Taylor series expansion, the functional value of the option at some point in time, T, is expressed as

$$C(S(T),T) = C(S(t),t) + C_s\big(S(T)-S(t)\big) + \frac{1}{2}C_{ss}\big(S(T)-S(t)\big)^2$$

$$+ C_t(T-t) + \frac{1}{2}C_{tt}(T-t)^2 + o(h)\,, \tag{1}$$

where C_s, C_{ss}, C_t, and C_{tt} are the first and second partial derivatives of the option with respect to *S*(*t*) and *t*, and $o(h)$ represents all the remaining higher order terms in the Taylor series expansion.

Expression (1) may be rewritten by defining the discrete increments of time and the stock price as $\Delta t \equiv (T-t)$ and $\Delta S(t) \equiv S(T)-S(t)$. Substituting these definitions into expression (1) yields

$$C(S(t)+\Delta S(t),t+\Delta t) = C(S(t),t) + C_s \Delta S(t) + \frac{1}{2}C_{ss}\left(\Delta S(t)\right)^2 + C_t \Delta t + \frac{1}{2}C_{tt}\Delta t^2 + o(h).$$

To obtain the infinitesimal change in the option price let $\Delta t \to 0$, which implies $\Delta S(t) \to 0$. The discrete increments are now expressed as infinitesimal changes denoted by *dt* and *dS*(*t*), thereby, changing the above expression to

$$C(S(t)+dS(t),t+dt) = C(S(t),t) + C_s dS(t) + \frac{1}{2}C_{ss}\left(dS(T)\right)^2 + C_t dt + \frac{1}{2}C_{tt}dt^2 + o(h). \tag{2}$$

Since *dt* and *dS*(*t*) represent infinitesimal changes in time and the stock price, the square and higher order terms are deemed negligible. That is, *dt* and *dS*(*t*) are infinitesimally small and their squares are even smaller. Moreover, the numbers become so negligible their impact to changes in the option price may be ignored. Ignoring the squared changes in time and the spot price reduces the above expression to

$$\begin{aligned} &C\big(S(t)+dS(t),t+dt\big) = C(S(t),t) + C_s dS(t) + C_t dt \\ &C\big(S(t)+dS(t),t+dt\big) - C(S(t),t) = C_s dS(t) + C_t dt, \\ &dC\big(S(t),t\big) = C_s dS(t) + C_t dt. \end{aligned} \tag{3}$$

Expression (3) is a well known result from calculus and it shows the total change in the derivative's price in terms of the infinitesimal and total changes of its determinants. Further, if analysts want to determine the value of the derivative security at some time *T*, they need only sum the individual increments. That is,

$$\begin{aligned} &\int_t^T dC\big(S(v),v\big) = \int_t^T C_s dS(v) + \int_t^T C_v dv,\ C\big(S(T),T\big) - C\big(S(t),t\big) = \int_t^T C_s dS(v) + \int_t^T C_v dv, \\ &C\big(S(T),T\big) = C\big(S(t),t\big) + \int_t^T \left[C_s dS(v) + C_v dv\right],\ C\big(S(T),T\big) = C\big(S(t),t\big) + \int_t^T dC\big(S(v),v\big). \end{aligned} \tag{4}$$

Expression (4) shows the value of the option at time *T* as the linear combination of its initial asset value and the summary of all the individual increments over a specified finite time horizon.

Stochastic Calculus: In the heuristic example above, the rules of calculus work well to describe the dynamics of an option when the option derives its value from deterministic variables. In practice, however, options derive their value from financial assets that are stochastic. In fact, the problem for all analysts is that they are interested in pricing options whose value is contingent on random variables measured over infinitesimal time intervals. Consider again, a call option written on a share of common stock at a particular point in time, but now let the stock price fluctuate randomly. To price the option, analysts would like to differentiate the relation between the call option and the stock price. In the previous example the rules of calculus worked well to describe this relation. However, the stock price is stochastic and the option is no longer a smooth function with respect to the stock price. Therefore, analysts may not use ordinary methods to derive a relation between the option and the stock price. Instead a new method to differentiate this relation must be used.

To model asset prices analysts use mathematics which assume that time passes continuously. This assumption is resourceful both mathematically and economically. From a modeling perspective, the advantage of continuous time mathematics is that it allows economists to obtain results that the discrete models cannot otherwise produce. Economically, the idea of continuous time fits the notion of markets and information. For instance, investors who are exposed to uncertainty, try to resolve any uncertainty in their investments by obtaining information which is continuously introduced to the market. While the arrival of news is random, it does provide feedback to investors who then buy and sell securities. As a result of this, trading markets witness instantaneous changes in asset prices and these prices may be viewed as a sequence of random variables measured over time. By definition this sequence of stock prices is called a stochastic process.

While stock prices tend to fluctuate randomly, this does not mean movements in prices are completely unpredictable. Investors continually gather information from a market in order to surmise what the value of an asset will be at some point in time. These forecasts are the investors' expectations about future changes in an asset's price given all current news. These expectations are not exact but do account for both expected and unexpected changes in prices over time.

To illustrate consider the change in a stock's price over a finite period of time expressed as

$$S(t+\Delta t)-S(t)=E_t\left[S(t+\Delta t)-S(t)\mid I_t\right]+\sigma\Delta Z(t)\,. \tag{5}$$

The first term on the right hand side is the expected movement in the stock price given all public information, I_t, up to time *t*. The second term is the unpredictable change in the stock price and is referred to as an innovation term. Expression (4) states that over time the stock price is expected to change by some known amount, $E_t\left[S(t+\Delta t)-S(t)\mid I_t\right]$, but the degree of certainty of this change is measured by $\sigma\Delta Z(t)$.

Price changes in Equation (5) are for discrete movements over a finite interval of time. To model this price behavior in continuous time, let $\Delta t \to 0$. Allowing the interval of time to approach zero captures the infinitesimal changes in the asset price, and this infinitesimal change is denoted as

$$dS(t)=\mu dt+\sigma dZ(t)\,, \tag{6}$$

where μ and σ are called the drift and diffusion coefficients and *dZ*(*t*) is the increment of a Brownian motion (also referred to as a Wiener process). Equation (6) is called a stochastic differential equation.

Uncertainty in the price dynamic, expression (6), is introduced by the last term, *dZ*(*t*). *dZ*(*t*) is called a Brownian motion and it is a Markov process that has been used in physics to describe the motion of a particle that is subject to a large number of infinitesimal shocks. For purposes of modeling price behavior analysts use this process to describe the motion of an asset's price that is subject to a large number of random news shocks. Important properties of the Brownian motion are:

(i) It is nowhere differentiable.

(ii) $Z(0) = 0$.

(iii) $dZ(t)$ has a normal distribution with mean 0 and variance dt for $s \le t$.

(iv) $Z(t_2) - Z(t_1), Z(t_3) - Z(t_2), \cdots, Z(t_n) - Z(t_{n-1})$ are independent for all $t_1 \le t_2 \le \cdots \le t_n$.

(v) $[dZ(t)]^2 = dt$ and $dZ(t)dt = 0$.

Property two states that the position of the process today is known given current information. Property three indicates that the price changes are normally distributed. Property four shows the prices follow a Markov property. That is, only the last observable price has any impact on forecasting the next increment. Intuitively, this property fits the notion that markets are semi-strong form efficient (all public and historical information is already incorporated in the asset price). Property five follows by construction since price changes are normally distributed.

Recall that economists want to differentiate the relation between the option and the stock price. Understanding the properties and dynamics of the stock price in expression (6) analysts can describe the impact that this price dynamic has on an option. For instance, recall the previous example where we found the dynamics of an option using a Taylor series expansion. Applying a Taylor series expansion to the option yielded

$$C(S(T),T) = C(S(t),t) + C_s\left(S(T) - S(t)\right) + \frac{1}{2}C_{ss}\left(S(T) - S(t)\right)^2$$

$$+ C_t(T-t) + \frac{1}{2}C_{tt}(T-t)^2 + o(h) \ , \qquad (7)$$

where C_s, C_{ss}, C_t, and C_{tt} are the first and second partial derivatives of $C\left(S(t),t\right)$, and $o(h)$ represents all remaining higher order terms of the Taylor series expansion. Next define the discrete increments in time and the stock price as $\Delta t \equiv (T-t)$ and $\Delta S(t) \equiv S(T) - S(t)$ and then substitute these definitions into expression (1) to obtain

$$C(S(t) + \Delta S(t), t + \Delta t) = C(S(t),t) + C_s \Delta S(t) + \frac{1}{2}C_{ss}\left(\Delta S(t)\right)^2 + C_t \Delta t + \frac{1}{2}C_{tt}\Delta t^2 + o(h).$$

Letting $\Delta t \to 0$, which implies $\Delta S(t) \to 0$, reduces the discrete increments to infinitesimal changes dt and $dS(t)$, thereby, changing the above expression to

$$C(S(t) + dS(t), t + dt) = C(S(t),t) + C_s dS(t) + \frac{1}{2}C_{ss}\left(dS(T)\right)^2 + C_t dt + \frac{1}{2}C_{tt}dt^2 + o(h). \qquad (8)$$

So far the Taylor expansion has been used in the same fashion as the previous example. The next step is to eliminate all negligible terms from expression (8). In the previous example, the underlying stock price $S(t)$ and time were deterministic and ordinary rules of calculus allowed the squared and higher order terms to vanish. However, in a stochastic environment the term $[dS(t)]^2$ does not vanish even though the higher order terms still vanish. Intuitively, $dS(t)$ in ordinary calculus is small such that $[dS(t)]^2$ is sufficiently close to zero. In a stochastic environment $dS(t)$ is a normally distributed random variable, which by definition means it has a positive variance. Therefore, $[dS(t)]^2$ cannot be removed from expression (8), but from property five of the Brownian motion, $[dS(t)]^2$ converges to dt and the higher order moments may be omitted. This leaves

$$C(S(t)+dS(t),t+dt)=C(S(t),t)+C_s dS(t)+\frac{1}{2}C_{ss}[dS(t)]^2+C_t dt\ .$$

Rearranging the above we have

$$C(S(t)+dS(t),t+dt)-C(S(t),t)=C_s dS(t)+\frac{1}{2}C_{ss}[dS(t)]^2+C_t dt\ ,$$

$$dC(S(t),t)=C_s dS(t)+\frac{1}{2}C_{ss}[dS(t)]^2+C_t dt\ . \tag{9}$$

Expression (9) is the well known result from stochastic calculus called Ito's lemma. Intuitively, Ito's lemma is the continuous time analog of the total derivative and it is the procedure that allows analysts to relate the dynamics of an underlying security to a corresponding derivative security.

Example of Ito's Lemma – Geometric Brownian Motion: Properties of geometric Brownian motion are discussed in Hull (2003) and Luenberger (1998). It is assumed the project's instantaneous value is defined by the following stochastic differential equation (SDE):

$$\frac{dS(t)}{S(t)}=rdt+\sigma dZ(t)\ , \tag{10}$$

where $S(t)$ is the underlying asset value, r is the drift term of the underlying asset, dt is the infinitesimal time change, σ is the volatility (or standard deviation) of the project's return, $dZ(t)$ is the increment of a standard Brownian motion. One objective in options pricing is to surmise the value of the security at a particular point in time. From the assumed dynamics, this entails finding a solution to the stochastic differential equation in expression (10).

To find a solution, let $H(t)=\ln S(t)$. The transformation and Ito's lemma (expression (9)) yield the following process for the increment of $H(t)$

$$dH(t)=H_s dS(t)+\frac{1}{2}H_{ss}[dS(t)]^2\ . \tag{11}$$

The process is expressed in terms of the stochastic differential for the spot price. To find a solution for $H(t)$ the analyst may substitute the expressions for the partial derivatives of $H(t)$ with respect to the stock price, the differential for the stock price $dS(t)$ and $[dS(t)]^2$ into expression (11). From equation (10), the squared increment of the spot price is found as follows

$$[dS(t)]^2=[rS(t)dt+\sigma S(t)dZ(t)]^2\ ,=r^2[S(t)]^2dt^2+\sigma^2[S(t)]^2[dZ(t)]^2+r\sigma[S(t)]^2dZ(t)dt\ .$$

The first term in the expression above is equal to zero. This is true since $dt^2 = 0$. Furthermore, the last term in the expression above is also zero since by definition of property five $dZ(t)dt = 0$. Therefore the above reduces to

$$[dS(t)]^2 = \sigma^2 [S(t)]^2 dt .$$

The partial derivatives of $H(t)$ with respect to $S(t)$ are

$$H_s(t) = \frac{1}{S(t)}$$

and

$$H_{ss}(t) = -\frac{1}{[S(t)]^2} .$$

Substituting the partials for $H(t)$, $dS(t)$ and $[dS(t)]^2$, into expression (11) yields

$$dH(t) = \left(\frac{1}{S(t)}\right) S(t)\left(rdt + \sigma dZ(t)\right) - \frac{1}{2}\frac{1}{[S(t)]^2}\left([S(t)]^2 \sigma^2 dt\right) = rdt + \sigma dZ(t) - \frac{1}{2}\sigma^2 dt ,$$
$$= \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dZ(t) .$$

To find the solution to expression (10), we integrate over the above expression to obtain

$$\int_t^T dH(v) = \int_t^T \left(r - \frac{1}{2}\sigma^2\right) dv + \int_t^T \sigma dZ(v) ,\ H(T) - H(t) = \int_t^T \left(r - \frac{1}{2}\sigma^2\right) dv + \int_t^T \sigma dZ(v) ,$$
$$H(T) - H(t) = \left(r - \frac{1}{2}\sigma^2\right)(T - t) + \int_t^T \sigma dZ(v)\ \ H(T) = H(t) + \left(r - \frac{1}{2}\sigma^2\right)\tau + \int_t^T \sigma dZ(v) .$$

Raising both sides to the power e yields

$$\exp\{H(T)\} = \exp\left\{H(t) + \left(r - \frac{1}{2}\sigma^2\right)\tau + \int_t^T \sigma dZ(v)\right\}\ \ S(T) = S(t)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau + \int_t^T \sigma dZ(v)\right\}$$
$$S(T) = S(t)\exp[Y(T)], \tag{12}$$

where $Y(T) = \left(r - \frac{1}{2}\sigma^2\right)\tau + \int_t^T \sigma dZ(v)$, and $\tau = T - t$ is the time to maturity. Expression (12) is a solution to equation (10) and it shows that the log returns, $\ln\left(\frac{S(T)}{S(t)}\right) = Y(T)$, for the spot price are normally distributed with a mean of $\left(r - \frac{1}{2}\sigma^2\right)\tau$ and a variance of $\sigma^2\tau$.

Since the returns are normally distributed, then by definition the spot price is log-normally distributed. That is if $Y(T)$ is normal then $\exp(Y(T))$ is lognormal. Therefore, the best forecast for the spot price at time T is found as

$$E_t[S(T)] = S(t)\exp\left\{E_t[Y(T)] + \frac{1}{2}V_t[Y(T)]\right\}$$
$$E_t[S(T)] = S(t)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau + \frac{1}{2}\sigma^2\tau\right\}$$
$$E_t[S(T)] = S(t)e^{r\tau}. \tag{13}$$

BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION DEVELOPMENT

The Black-Scholes pricing equation is stated in most financial option textbooks, however, the mathematical details are rarely presented. Pedagogical references include Hull (2003), Luenberger (1998), Wilmott et al. (1995), and Neftci (2000). Additionally, the focus of these textbooks is directed to the Black-Scholes result as opposed to its development. When the textbooks illustrate the derivation of the Black-Scholes pricing equation, the derivation is filled with mathematical 'shortcuts' and convenient assumptions which often leaves the reader lacking any economic intuition. In our analysis, we focus on the economics and the mathematics behind the Black-Scholes equation.

The presentation for the Black-Scholes equation is segmented into three stages. The first phase consists of developing the dynamics of the call option using the mathematics from section two. The second phase focuses on constructing an equilibrium pricing condition by strategically combining the stock, the option, and a risk-free asset in a portfolio. The last phase solves for the Black-Scholes solution from the equilibrium condition.

Dynamics: Black and Scholes (1973) posit that the stock prices follow an exogenously determined stochastic process (geometric Brownian motion), which we formally describe as

$$\frac{dS(t)}{S(t)} = \mu_s dt + \sigma_s dZ(t), \tag{14}$$

where μ_s is the mean return σ_s is the diffusion coefficient, and $dZ(t)$ is the increment of a standard Brownian motion. Equation (14) shows that stock price returns appreciate over time by some amount μ_s, but are also influenced over time by some uncertainty measure, σ_s. In the Black-Scholes model an option contract derives its value from an underlying stock, whose price obeys the dynamics in expression (14). Since the option is a function of a stochastic process (the stock price) the option itself is a stochastic process. If an option contract can be written as a twice-continuously differentiable function of the stock price and time, namely $C(S,t)$, then the option return dynamics can be written in a similar form as

$$\frac{dC(S,t)}{C(S,t)} = \mu_c dt + \sigma_c dZ(t), \tag{15}$$

where μ_c is the mean return of the option, σ_c is the diffusion coefficient of the option, and $dZ(t)$ is the increment of a standard Brownian motion. Equation (15) is only a general expression for the return dynamics of the option contract where μ_c and σ_c have not been formally defined. The actual drift and diffusion terms for the option may be determined by formally developing the stochastic differential equation in expression (15). This is done using Ito's lemma.

We define the option value as $C(S,t)$. Invoking Ito's lemma the increment for the option contract is

$$dC(S,t) = C_s dS(t) + \frac{1}{2}C_{ss}[dS(t)]^2 + C_t dt,$$

$$= C_s[\mu_s S(t)dt + \sigma_s S(t)dZ(t)] + \frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2 dt + C_t dt,$$

$$= \left[\frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2 + \mu_s S(t)C_s + C_t\right]dt + \sigma_s S(t)C_s dZ(t). \quad (16)$$

To express these movements in terms of returns we divide the left-hand side and the right-hand side by $C(S,t)$. That is,

$$\frac{dC(S,t)}{C(S,t)} = \left[\left(\frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2 + \mu_s S(t)C_s + C_t\right)\Big/C(S,t)\right]dt + [\sigma_s S(t)C_s/C(S,t)]dZ(t) \quad (17)$$

Comparing expressions (17) and (15), an analyst can identify that both are expressed in the same units (returns) and both expressions describe the same time series behavior for the option contract. Hence, they are equal and we can equate the drift and diffusion terms in both expressions. As such,

$$\mu_c = \left[\frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2 + \mu_s S(t)C_s + C_t\right]\Big/C(S,t), \quad (18)$$

$$\sigma_c = \sigma_s S(t)C_s/C(S,t). \quad (19)$$

In addition to the coefficients above, the reader probably noted that we did not formally define the innovation term $dZ(t)$ in expression (15). We now see the Brownian motions in equations (15) and (17) are the same as the Brownian motion term in the spot price shown in equation (14). Thus, erratic price movements in the call option and the stock originate from the same source.

<u>Arbitrage Strategy</u>

Given the time series behavior of the assets, investors would like to combine the stock, the option, and the risk free asset in a portfolio so the payout next period is known with certainty (Merton (1973)). To illustrate, consider an investor who decides to hold portions of the stock, the option, and a riskless asset in a portfolio, where the aggregate investment in the portfolio is zero. The weights are denoted as $w_1, w_2,$ and w_3 respectively. By definition the portfolio weights sum to one, $\sum_{j=1}^{3} w_j = 1$, and the value of the portfolio is denoted as $A(t)$. Since the portfolio is a function of both the stock and the option, an analyst may express the return process of the portfolio as a stochastic process. This dynamic is expressed in a similar fashion as the stock return dynamics and is denoted as,

$$\frac{dA(t)}{A(t)} = \mu_A dt + \sigma_A dZ(t)\,, \tag{20}$$

where μ_A is the mean return and σ_A is the diffusion coefficient. For now we define $dZ(t)$ as the increment of a standard Brownian motion, with an expected value of zero and a variance of $\sqrt{t}$. Below we will see that this term is the linear combination of the individual asset diffusion coefficients.

Since the portfolio is a linear combination of the three assets, its follows from modern portfolio theory that the drift term for the portfolio is the linear combination of the drift coefficients of the individual assets in the portfolio. This is,

$$\mu_A = w_s\mu_s + w_c\mu_c + w_c\mu_r\,. \tag{21}$$

In addition, portfolio theory shows the variance of a portfolio is equal to the sum of the variances and covariances of all assets in the portfolio. Formally this is

$$\sigma_A^2 = w_s^2\sigma_s^2 + w_c^2\sigma_c^2 + w_r^2\sigma_r^2 + 2w_s w_c Cov(S,C) + 2w_s w_r Cov(S,r) + 2w_c w_r Cov(C,r) \tag{22}$$

Given the variance and return for the portfolio, the objective for any investor is to find a portfolio weighting scheme that minimizes the variance while maintaining a positive return. Formally we may define the problem as such:

$$Min\,\sigma_A^2 = w_s^2\sigma_s^2 + w_c^2\sigma_c^2 + w_r^2\sigma_r^2 + 2w_s w_c Cov(S,C) + 2w_s w_r Cov(S,r) + 2w_c w_r Cov(C,r)$$

subject to

$$\mu_A = w_s\mu_s + w_c\mu_c + w_c\mu_r = \theta\,,$$

and

$$\sum_{j=1}^{3} w_j = 1\,.$$

Inspection of the objective function shows the interest rate is nonstochastic, therefore, σ_r, $Cov(S,r)$, and $Cov(C,r)$ are all equal to zero. As for the underlying asset and option, both are influenced by the same innovation term which by definition indicates they are perfectly correlated. From basic statistics the covariance between the option and the stock is expressed as

$$Cov(S,C) = \sigma_s\sigma_c\rho_{sc}\,,$$

where ρ_{sc} is the correlation coefficient. In this particular case $\rho_{sc} = 1$, which yields

$$Cov(S,C) = \sigma_s\sigma_c\,.$$

Substituting this into the expression for the portfolio variance yields

$$\sigma_A^2 = w_s^2\sigma_s^2 + w_c^2\sigma_c^2 + 2w_s w_c\sigma\sigma_c\,.$$

This expression can be expressed as

$$\sigma_A^2 = \left(w_s\sigma_s + w_c\sigma_c\right)^2.$$

Taking the square root of this expression yields

$$\sigma_A = w_s\sigma_s + w_c\sigma_c \tag{23}$$

Thus, the diffusion of the portfolio is a linear combination of the diffusion for the underlying asset, σ_s, and the option, σ_c.

From inspection of expression (23), the objective function will be minimized when the standard deviation to the portfolio is equal to zero. One possible solution for this optimization problem is the trivial solution, $w_j = 0$ for j = (s, c, r). However, from the constraint, $\sum_{j=1}^{3} w_j = 1$, this is not permissible. To find an alternative solution, consider expressions (21) and (23). Investors wish to make their portfolio variance zero, which implies

$$\sigma_A = w_1\sigma_s + w_2\sigma_c = 0.$$

That is investors wish to find a portfolio that yields a riskless return. Consequently, if the portfolio in expression (20) is risk-free, then the return to this portfolio over the investment horizon should equal the riskless rate of return, $\mu_A = r$. Otherwise arbitrage possibilities exist. Using expression (21) and the portfolio weighting constraint, $\sum_{j=1}^{3} w_j = 1$, the portfolio return may be written as

$$\mu_A = w_s(\mu_s - r) + w_c(\mu_c - r) + r = r \tag{24}$$

Rearranging expression (24) and considering the portfolio variance, investor wish to find a nontrivial solution for the following system of homogeneous equations

$$\mu_A - r = w_1(\mu_s - r) + w_2(\mu_c - r) = 0 \tag{25}$$
$$\sigma_A = w_1\sigma_s + w_2\sigma_c = 0. \tag{26}$$

Using expression (25), the weight for stock is obtained and is equal to

$$w_1^*(\mu_s - r) + w_2^*(\mu_c - r) = 0,$$

$$w_1^*(\mu_s - r) = -w_2^*(\mu_c - r),$$
$$w_1^* = -w_2^*\frac{(\mu_c - r)}{(\mu_s - r)}.$$

Substituting this expression into expression (26) yields

$$\left(-w_2^*\frac{(\mu_c-r)}{(\mu_s-r)}\right)\sigma_s+w_2^*\sigma_c=0\,,$$

$$w_2^*\left[\left(-\frac{(\mu_c-r)}{(\mu_s-r)}\right)\sigma_s+\sigma_c\right]=0\,.$$

The goal is to find the non-trivial solution $w_i^*\neq 0$. The only way the equation above equals zero is if the expression inside the brackets equals zero. This implies

$$\left(-\frac{(\mu_c-r)}{(\mu_s-r)}\right)\sigma_s+\sigma_c=0\,,$$
$$\left(\frac{(\mu_c-r)}{(\mu_s-r)}\right)\sigma_s=\sigma_c\,,$$
$$\sigma_s(\mu_c-r)=\sigma_c(\mu_s-r)$$
$$\frac{(\mu_s-r)}{\sigma_s}=\frac{(\mu_c-r)}{\sigma_c}\,. \tag{27}$$

Expression (27) is an equilibrium condition that must hold under a no-arbitrage constraint in order for an optimal weighting scheme to exist. In equilibrium, expression (27) illustrates that all assets earn the same the reward to risk ratio, $\frac{(\mu_i-r)}{\sigma_i}$. Intuitively, the market is pricing risks for all assets in the same manner. Using the equilibrium condition in expression (27), we may obtain the expected dynamics of the option contract. Substituting the expression for the option's drift and diffusion coefficients (expressions (18) and (19)) into the equilibrium condition lead to:

$$\frac{\mu_s-r}{\sigma_s}=\frac{\left\{\left[\frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2+\mu_sS(t)C_s+C_t\right]\Big/C(S,t)\right\}-r}{\sigma_s\;S(t)C_s/C(S,t)}\,,$$

$$\frac{(\mu_s-r)\sigma_sS(t)C_s}{\sigma_sC(S,t)}=\frac{\left[\frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2+\mu_sS(t)C_s+C_t\right]}{C(S,t)}-r\,,$$

$$\frac{(\mu_s-r)\sigma_sS(t)C_s}{\sigma_s}=\frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2+\mu_sS(t)C_s+C_t-rC(S,t),$$

$$(\mu_s-r)S(t)C_s=\frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2+\mu_sS(t)C_s+C_t-rC(S,t),$$

$$-\mu_sS(t)C_s+\mu_sS(t)C_s+\frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2+rS(t)C_s+C_t-rC(S,t)=0\,,$$

$$\frac{1}{2}C_{ss}\sigma_s^2[S(t)]^2+C_srS(t)-rC(S,t)+C_t=0\,. \tag{28}$$

Expression (28) is a partial differential equation for an option contract written on a stock whose price follows a geometric Brownian motion. The result is unique in that the price dynamics of the option contract are now expressed deterministically. This is made possible from the investor's ability to construct a self-replicating arbitrage portfolio of the stock, option, and risk-free asset. Notably, the use of this arbitrage portfolio is the continuous-time analog to the discrete-time binomial model discussed in Cox et. al. [3] and Appendix A.

Black-Scholes Solution

Using standard solution techniques analysts may determine the value of an option contract from expression (28) either analytically or numerically. To derive the value of the option the analyst only needs to specify the necessary boundary conditions to solve the partial differential equation. For the Black-Scholes model the boundary conditions for the option are

$$C(0,\tau) = 0\,, \tag{29}$$

$$C(S,0) = \max[0, S(T) - X], \tag{30}$$

where X is the exercise price of the option and $\tau = T - t$ is the time to maturity. Intuitively, expressions (29) and (30) are contractual clauses for an option contract. Expression (29) implies that if a market does not exist for the underlying asset the option is worthless. Expression (30) states that at maturity the value of the option will equal the greater of the two amounts, $S(T) - X$ or 0. The function (solution) that satisfies (28), (29), and (30) simultaneously is

$$C(S,t) = S(t)N(d_1) - e^{-r\tau} XN(d_2)\,, \tag{31}$$

where

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{z^2}{2}} dz\,, \tag{32}$$

$$d_1 = \left[\ln\left(\frac{S(t)}{X}\right) + \left(r + \frac{1}{2}\sigma_s^2\right)\tau \right] \Big/ \sigma_s \sqrt{\tau}\,, \tag{33}$$

$$d_2 = d_1 - \sigma_s \sqrt{\tau}\,. \tag{34}$$

Expression (31) is the well known Black-Scholes pricing formula for a call option. The expression is a function of the underlying stock price, the exercise price, the volatility of the stock price, the risk free rate of interest, and time to maturity.

CONCLUSIONS

The main contribution of this paper is the detailed mathematical and economic account of the Black-Scholes development. The paper begins by introducing the concept of stochastic versus traditional calculus and then develops an expression for geometric Brownian motion. The main portion of the paper details the replicating portfolio argument of financial option pricing, the development of the economic equilibrium reward-to-risk ratio, and the Black-Scholes-Merton ordinary differential equation. To fully comprehend the literature and to acquire an appreciation for the modeling techniques, a basic understanding of financial option pricing mathematics is a necessary prerequisite.

APPENDIX

Appendix: Binomial Example

Consider a simple discussion of Cox, Ross, and Rubinstein's (1979) binomial lattice option valuation technique [3]. Technically, their approach is a numerical approximation to its Black-Scholes counterpart. However, the principles demonstrated are key to truly understanding option-pricing. In general, the approach assumes: (1) the underlying asset follows a discrete, binomial, multiplicative stochastic process throughout time, (2) arbitrage-free pricing, and (3) the law of one price, which states that if two portfolios are equal in value at the expiration time *T*, then they must have equivalent values today. Using these assumptions, a portfolio consisting of the underlying asset and risk-free bonds may be formed that replicates the option payoff in any state of nature. This portfolio will consist of Δ shares of the underlying asset financed in part by an amount \$*b* at the risk free rate. Figure 1 demonstrates the replicating portfolio concept.

Figure 1. Replicating portfolio option valuation approach

$$C_0 = \Delta S_0 + b \quad \begin{cases} C_u = \Delta S_u + b(1+r) \\ C_d = \Delta S_d + b(1+r) \end{cases}$$

Time: $t \longrightarrow T$

Δ = number of shares of the underlying asset
b = amount of cash borrowed at the risk-free rate
r = risk free rate of interest
S_0 = value of the underlying asset today
S_u = upward movement value of the underlying asset in the future at time *T*
S_d = downward movement value of the underlying asset in the future at time *T*
C_0 = value of the call option today
$C_u = \max(S_u - I, 0)$
$C_d = \max(S_d - I, 0)$

The motivation of the investor is to construct the portfolio so that the option payoff at any future time is known today. Under the assumption of the law of one price, the cost to set up the replicating portfolio must be equal to the option's value today. Solving the equations for C_u and C_d in Figure 1 yields:

$$\Delta = \frac{C_u - C_d}{S_u - S_d} \qquad \text{and} \qquad b = \frac{uC_d - dC_u}{(1+r)(u-d)}$$

The value of the option today, C_0, is then:

$$C_0 = \Delta S_0 + b = \frac{C_u - C_d}{S_u - S_d} S_0 + \frac{uC_d - dC_u}{(1+r)(u-d)} = \frac{C_u - C_d}{S_0(u-d)} S_0 + \frac{uC_d - dC_u}{(1+r)(u-d)}$$

$$= \frac{1}{(1+r)} \left(\frac{(1+r)-d}{u-d} C_u + \frac{u-(1+r)}{u-d} C_d \right)$$

Defining $p = \frac{(1+r)-d}{u-d}$ as the synthetic (or risk-neutral) probability, the option price may be stated as:

$$C_0 = \frac{1}{(1+r)}\left(pC_u + (1-p)C_d\right)$$

The option value today is the discounted expected payoff using the risk-free rate of interest and risk-neutral probabilities.

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