INTRAPORTFOLIO CORRELATION: AN APPLICATION FOR INVESTMENTS STUDENTS

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ABSTRACT

Intraportfolio correlation (IPC), a measure of portfolio diversification, is becoming increasingly popular among investment practitioners. However, despite the assertions of these adherents, IPC is far from a free lunch. Instead, it is a simplistic and flawed measure that ignores material information about the relationships among portfolio assets. Deconstructing the IPC therefore can be a productive and educational exercise (and a cautionary tale) for students of portfolio theory. In this paper, we describe IPC and offer suggestions for incorporating it into an introductory investments course.

JEL: G10, G11

KEYWORDS: Portfolio Theory, Diversification, Finance Pedagogy

INTRODUCTION

Introductory courses on investments are almost certain to cover the portfolio theory of Markowitz (1952) and the Capital Asset Pricing Model of Sharpe (1964), Lintner (1965), Mossin (1966), and Treynor (1961). However, the theory students tackle in the classroom may have morphed into something quite different by the time they encounter it in the "real world."In this paper, we consider one such transformation: a metric called "intraportfolio correlation" (IPC, or "Q"), a "weighted average intra-portfolio correlation that translates the range of correlations to percentage values" (Damschroeder, 2010). It is a weighted, standardized correlation coefficient. This measure has recently found favor with some practitioners, who assert that it easily quantifies a portfolio's diversification. However, confounding these claims are inconsistencies in various definitions of IPC and apparent confusion about basic mechanics of portfolio variance. These inconsistencies invite critical examination by investments students.

We discuss the claims made for intraportfolio correlation, then describe ways instructors can incorporate investigation of those claims into basic investments courses. The paper proceeds as follows. In the next section, we briefly review the literature underlying traditional approaches to teaching portfolio theory in introductory investments courses. We then turn to the intraportfolio correlation measure, first defining it, then exploring its applicability to two- and three-asset portfolios. Finally, we propose some ways that students could enhance their understanding of portfolio theory by evaluating the IPC.

LITERATURE REVIEW

Our goal is to demonstrate how investments instructors can incorporate an analysis of Q into the traditional curriculum. Thus, we are concerned with the concepts making up this curriculum and with the pedagogical approaches to teaching it. In this section, we consider both. We will show that, on the conceptual side, Q integrates easily into discussions of portfolio theory and market efficiency; as for pedagogy, Q fits seamlessly into applied spreadsheet projects. Markowitz (1952) derived the portfolio theory concepts that underlie the curriculum in investments courses. Students of portfolio theory—and therefore students of Q—are by definition students of Markowitz, and, while they may not recognize his notation, students should be very familiar with his development of portfolio expected return and variance. He starts by defining the mean and variance for random variables, moves to weighted sums of those variables (i.e., "[t]he expected value of a weighted sum is the weighted sum of expected values") and to the definitions of covariance and correlation, then finishes by identifying efficient portfolios. This has become the standard development in investments courses.

The distillation of Markowitz's work into textbooks has become so standardized, in fact, that serious students would benefit from reconciling the differences between the modern treatments and Markowitz's original graphs. For example, textbooks routinely illustrate the efficient set of risky assets as a parabola in [σ_p, $E(R_p)$] space. Markowitz presents the same concept first with these axes reversed—with expected return on the *x* axis, so that the efficient set curves up from the southwest to the northeast—and then, more interestingly, on $weight_1$ and $weight_2$ axes. In the latter case, he is presenting results for three-asset portfolios, and the efficient set is the line traced out by the tangencies of isomean lines and isovariance curves. (See also Tobin, 1958.) Markowitz's graphs underscore points obscured in the basic parabola graphs: the dependence of both $E(R_p)$ and σ_p on the portfolio weights, rather than of $E(R_p)$ directly on σ_p; the elliptical nature of the isovariance curves; the linearity of the efficient set in the [w_1, w_2] space. We use this type of graph when discussing three-asset portfolios and Q, in a later section.

Markowitz's portfolio solutions were revolutionary, but were also extremely difficult to *use*. Identifying the efficient set requires quadratic programming, since portfolio variance is not a linear function of the assets' weights; this was a decidedly complex task in the computing environment of the 1960s and 1970s. Practitioners and academics soon began searching for an efficient, simple algorithm for identifying efficient portfolios. (Rubenstein, 2006, summarizes much of this research.) Q is just a very recent entrant into a crowded market of basic approximations to the efficient frontier. Understanding the earlier methods will help us evaluating Q's contribution.

Sharpe's single index model is the foundation for many of the early approximations. This model assumes that the covariance between the returns of a pair of assets is explained only by their individual relationships with a third factor, significantly simplifying the determination of the covariance matrix. For example, Sharpe himself (1967) suggests a linear approximation for portfolio variance, based on the single index model. (See also Sharpe, 1971.) Since mutual funds may hold no more than 5% of their portfolios in any single security, he proposes that asset weights are small enough to justify a linear estimate. His empirical results demonstrate a good fit: "For any given average rate of return, the portfolio selected using linear approximation had variance less than 1% greater than that of the most efficient portfolio (i.e., the one selected using the full covariance matrix)." The results were especially good for more aggressive portfolios (according to Sharpe, expected return—which is exact, not approximated—is relatively more important than variance for these portfolios). Despite the fit of this linear approximation, the search for quick solutions to Markowitz's equations continued.

To derive closed-form expressions for the optimum portfolio's assets and weights, Elton, Gruber, and Padberg (1976) start with Sharpe's single index model (as do Treynor and Black, 1973), then assume that all assets' correlations are the same and that a risk-free asset exists. Their "simple" solutions do not require parametric quadratic programming. As we will see, they nonetheless are vastly more complex than Q. Unlike Elton, Gruber, and Padberg's solution, Q ignores borrowing/lending possibilities, the distinction between market and unique risk, the possibility for short selling, and the identification of the available security set. It ignores return. It takes the correlation matrix as given. Combining Q with other metrics, as GSphere does, admits some of these defects. However, Elton, Gruber, and Padberg's equations clarify at least two things for students of Q: first, the search for simple portfolio solutions is as old as portfolio theory itself; and second, even "simple" solutions in highly stylized scenarios will be vastly more complex than intraportfolio correlation. Markowitz's diversification results lead investments students to ideas of market efficiency. How do his insights affect our understanding of the potential for active management? Since Q is marketed as a portfolio-evaluation tool, it is a useful addition to discussions of efficiency, and the extensive literature in this area can help us appreciate its contribution.

Treynor and Black (1973) is an early example of this literature. These authors consider the incorporation of active management into the "essentially objective, statistical approach to portfolio selection of Markowitz" and Sharpe (1964). They propose a tripartite model of returns, modeling return as the sum of a riskless component, a premium for market exposure, and an "independent return." They call the expected value of the independent return the "appraisal premium"; it represents the return to the actively managed part of the portfolio.

Given this framework, Treynor and Black describe an approach in which managers act on their unique information and insights about stocks, accumulating an active position. Managers then adjust for the market risk of this position by buying or selling the market portfolio in the passive part of their portfolio. In this world, a "perfectly diversified" portfolio is one with no nonmarket risk. However, such a portfolio is not risk-free, and, more importantly, may not be "optimally balanced." Instead, an optimally balanced portfolio takes full advantage of opportunities to accept asset-specific risk profitably: "any improvement in the quality of security analysis, or in the number of securities analyzed at a given level of quality, can only cause an optimally balanced portfolio to become less well diversified... In general it is not correct to assume that optimal balancing leads either to negligible levels of appraisal [active] risk or to negligible levels of market risk." Thus, Treynor and Black (1973) suggest that effective active managers would have no use for a metric like Q. For decades, finance professors have built investments courses around the concepts of market efficiency and portfolio theory, based on the literature we have just discussed. Only recently, however, have they been able to have students explore these ideas using sophisticated spreadsheet applications. We turn now to the relatively recent pedagogical literature outlining these sorts of projects. Students who have done a spreadsheet project—who can optimize, run regressions, and perform simulations—are unlikely to see the need for a simplistic metric like Q.

There are many examples of investments projects incorporating spreadsheets. For example, Kalra and Weber (2004) outline a basic task-based investments project covering the standard metrics for a single stock. Kish and Hogan (2009) present an expanded project allowing students to practice basic portfolio theory using multiple assets. These basic approaches are easy to extend. For example, Neumann (2008) motivates his project using the *Wall Street Journal's* long-running dartboard contest, and Girard, Pondillo, and Proctor (2005) incorporate performance attribution analysis. For more academically rigorous courses, Carter, Dare, and Elliott (2002) show how students can find mean-variance efficient portfolios using Excel's Solver; Johnson and Liu (2005) extend this procedure to allow for short sales. Pushing the use of technology even farther, the most recent papers incorporate Monte Carlo simulations. For example, Ammar, Kim, and Wright (2008) demonstrate simulations using both Excel's built-in functions and Crystal Ball add-ins. The resulting histograms—the products of many hundreds of trials on many different portfolios—clarify the abstract concept of portfolio risk. These sorts of projects and applications are becoming the norm in investments classrooms. It is to students in these modern courses that we wish to introduce intraportfolio correlation.

Defining The Intraportfolio Correlation

According to adherents, intraportfolio correlation *defines* diversification (see, for example, Hedge Funds Consistency Index, http://www.hedgefund-index.com/d_diversification.asp). However, there are competing definitions for IPC, which complicates evaluation. One definition that comes up frequently is:

$$Q = \sum_i \sum_j w_i w_j \rho_{ij} , \qquad (1)$$

$i \neq j$, where w_i is the fraction of the portfolio invested in asset i, w_j is the fraction of the portfolio invested in asset j, and ρ_{ij} is the correlation between assets i and j. (See, for example, EconomicExpert.com and Gravity Investments' "GSphere" optimization platform. Interestingly, Gravity Investments says that it "invented and patented diversification optimization, measurement, search and visualization.") Q, the intraportfolio correlation, then feeds into a calculation for the "percent of diversifiable risk removed":

$$\% \text{ of diversifiable risk removed} = \frac{(1-Q)}{2} \tag{2}$$

For example, if Q = 1 (its highest possible value), diversification is zero, while if Q = -1 (its lowest possible value), diversification is perfect. However, there is a problem with this definition. Consider an equally weighted portfolio of two assets *i* and *j*, where $\sigma_i^2 = \sigma_j^2$. If assets *i* and *j* are perfectly negatively correlated, this portfolio would have a Q of $2{*}(.5)^2{*}(-1) = -.5$, implying that the percent diversified is (1+.5)/2 = .75, or 75%. However, in fact, this equally weighted portfolio actually would *eliminate* risk. Therefore, we will use an alternative definition of intrapoprtfolio correlation (found, for example, at Hedge Fund Consistency Index, 2011 and WordIQ, 2011):

$$Q = \frac{\sum_i \sum_j w_i w_j \rho_{ij}}{\sum_i \sum_j w_i w_j} \tag{3}$$

$$= [W'W]^{-1} W'[R - \text{Diag}(R)]W,$$

where W is the *n*x1 vector of weights, R is the *n*x*n* correlation matrix, and Diag(•) is the *n*x*n* matrix of the diagonal elements of the target matrix. Using (3), Q equals -1 for the equally weighted equal-variance portfolio, so that the percent diversified is 100%, as it should be. Unfortunately for Q adherents, defining the metric to be consistent with (2) does not mean that Q is actually helpful. For our two-asset, perfectly negatively correlated portfolio, for example, Q will *always* equal -1, even though portfolio variance, σ_p^2, is zero only when the assets are weighted equally. Q always sends the same signal, so it is not providing an adequate measure of true diversification. In the next section, we will examine these sorts of problems with Q in the context of two-asset portfolios; we will then go on to consider them in the more generalizable case of three assets.

Q And Two-Asset Portfolios

The two-asset case is an important starting point for students of portfolio theory. It is not just that the math is most tractable for this case; it is that two-asset portfolios have some interesting characteristics. Unfortunately for students, however, Q will not help them appreciate these features. Two-asset portfolios are unique in that all two-asset portfolios are minimum-variance—they all offer the lowest-variance way to deliver a given expected return. That is, there is only one combination of weights that delivers a specified expected return, since:

$$w_i = \frac{[E(R_p) - E(R_j)]}{[E(R_i) - E(R_j)]}, \tag{4}$$

(where E(R) is expected return), and $w_j = (1 - w_i)$. However, while all two-asset portfolios are minimum-variance (having the lowest variance for a given expected return), they are not all efficient (having the highest expected return for a given variance). To find the efficient set, we first find the portfolio that has the global minimum variance; portfolios that lie below this point (in [σ, E(R)] or [σ^2, E(R)] space) are inefficient. We can easily find the global minimum-variance portfolio. Differentiating the portfolio variance equation with respect to w_i, we find that $d\sigma_p^2/dw_i = w_i*[\sigma_i^2 + \sigma_j^2 - 2*\sigma_{ij}] - \sigma_j^2 + \sigma_{ij}$, where σ_{ij} is the assets' covariance. Setting this equal to zero and solving for w_i, we find that we can minimize portfolio variance by setting w_i as:

$$w_i = \frac{\sigma_j^2 - \sigma_{ij}}{\sigma_i^2 + \sigma_i^2 - 2*\sigma_{ij}}. \qquad (5)$$

(See, for example, Martin, Cox, and MacMinn, 1988, Chapter 8, and Bodie, Kane, and Marcus, 1993, Chapter 7.) Now, equating (4) and (5), we can find the expected return for the global minimum-variance portfolio as:

$$E(R_p)_{min\text{-}var} = E(R_j) + \frac{\sigma_j^2 - \sigma_{ij}}{\sigma_i^2 + \sigma_j^2 - 2*\sigma_{ij}} * [E(R_i) - E(R_j)]. \qquad (6)$$

As equations (5) and (6) make clear, the variance of a two-asset portfolio depends upon the weighting scheme chosen, and there is only one *global* minimum-variance portfolio. However, the dependence of σ_p^2 on w_i is obscured by the intraportfolio correlation measure. In fact, Q is invariant to w_i in the two-asset case, as we can see in the equation below, and in Figure 1:

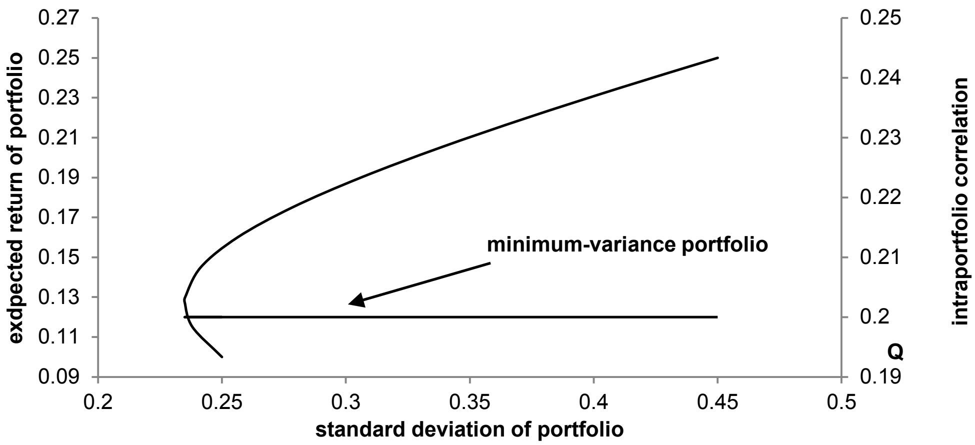
$$Q = \frac{2w_i w_j \rho_{ij}}{2w_i w_j} = \rho_{ij}. \qquad (7)$$

All Q tells us is the correlation coefficient—which we already knew—not the amount of realized diversification in a specific portfolio. (This can also be true for the *n*-asset case. For example, in their approximation, Elton, Gruber, and Padberg, 1976, assume of equal correlations for all asset pairs—an assumption they say "produces better estimates of future correlation coefficients than do historical correlation coefficients or those produced from the single index approximation." Given this assumption, Q again equals ρ.) By returning us to the correlation coefficient, Q simply reminds us of the *potential* for diversification. It certainly does not, as proponents such as Gravity Investments claim, lead us directly to the "percent of diversifiable risk removed. "Of course, it is difficult to imagine what a concept like "percent of diversifiable risk removed" even means within a two-asset context. Given that all portfolios are minimum-variance, every expected return is delivered with the lowest possible risk, given ρ_{ij}. However, it might be instructive for students to try to develop their own interpretation. For example, given that Q = ρ, perhaps its boosters are looking for a measure more like the following:

$$\text{relative diversification effectiveness} = \frac{[\sigma_p \mid \rho_{ij} = 1] - [\sigma_p \mid \rho_{ij}]}{[\sigma_p \mid \rho_{ij} = 1] - [\sigma_{ij} \mid \rho_{ij} = -1]}. \qquad (8)$$

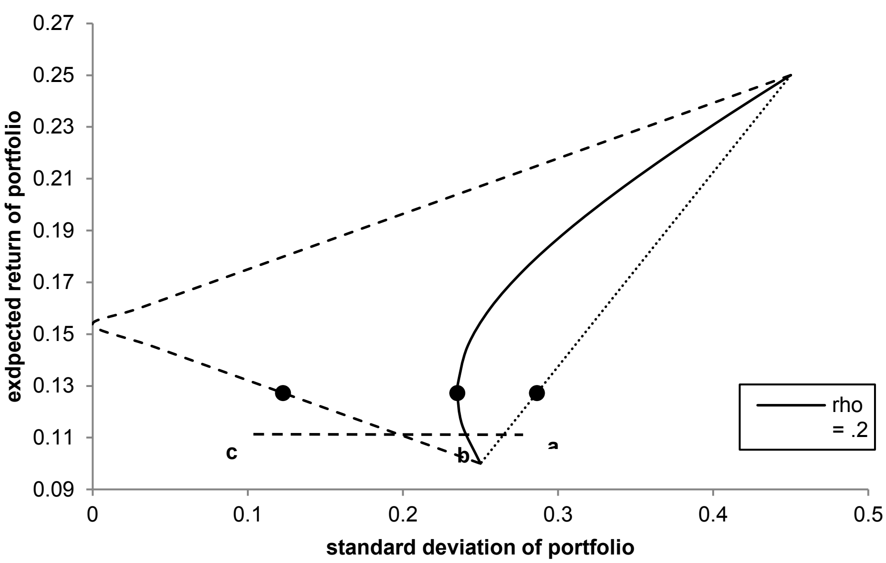
That is, for any level of expected return, how much diversification does the portfolio offer, given the actual correlation between the assets, relative to the worst-case correlation scenario (ρ_{ij} = 1) and the best-case scenario (ρ_{ij} = -1)? Figure 2 illustrates the concept, which we will call Z.

Figure 1: Q in the Standard Two-Asset Case



This figure shows the standard [σ_p, $E(R_p)$] bullet-shaped relationship for the two-asset case (assuming $E(R_i)$ = 10%, σ_i = 25%, $E(R_j)$ = 25%, σ_j = 45%, and ρ_{ij} = .2). Note that Q does not change as the relative weights of the two assets change: it always equals ρ .

Figure 2: Correlation and Diversification Potential in the Two-Asset Case



This figure illustrates the diversification potential from various values of ρ_{ij}. The solid curve is the same as that from Figure 1, and assumes a correlation of 0.2. The dotted line assumes that the two assets are perfectly positively correlated; the dashed lines assume perfect negative correlation. Points a, b, and c all lie at the same level of $E(R_p)$;point b is the global minimum-variance portfolio from Figure 1. Equation (8) measures the distance from a to b, relative to the distance from a to c: (a-b)/(a-c). Larger values for this ratio imply more effective diversification.

Since the standard deviation for a two-asset portfolio is a simple linear combination of the assets' standard deviations when $\rho_{ij} = 1$ ($\sigma_p = \sigma_j + w_i*[\sigma_i - \sigma_j]$) or when $\rho_{ij} = -1$ ($\sigma_p = w_i*[\sigma_i + \sigma_j] - \sigma_j$), we can rewrite (8) as:

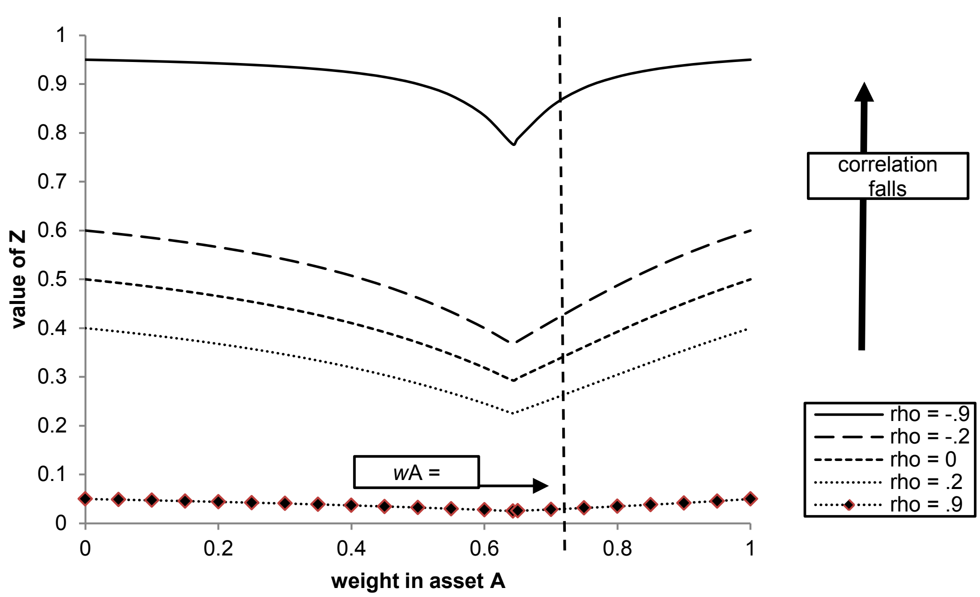
$$Z = \frac{\sigma_j + w_i*(\sigma_i - \sigma_j) - \sigma_p}{2*\sigma_j*(1-w_i)}, \tag{9}$$

for $1 \geq w_i \geq \sigma_j/(\sigma_i+\sigma_j)$. (These are the weights of asset *i* for which the $\rho = -1$ relationship is shown as a line segment with a positive slope. For these weights, the σ_p expression for $\rho = 1$ is positive. There is a comparable simplified version of (8) corresponding to the weights generating the negatively sloped line segment.) The behavior of Z is shown in Figure 3. Note that Z rises as the assumed correlation between the two assets falls. As shown in Figure 3, Z is minimized when $w_i = \sigma_j/(\sigma_i+\sigma_j)$: that is, when the $\rho = -1$ curve hits the *y* axis (so that $\sigma_{p|\rho=-1} = 0$). It is maximized as w_i or w_j approach 1:

$$\lim Z_{wi \to 1} = \lim Z_{wi \to 1} = \frac{1-\rho_{ij}}{1-(-1)} = \frac{1-\rho_{ij}}{2}. \tag{10}$$

Note that (10) is equivalent to (2) in this case: since Q = ρ when there are 2 assets, the "percent of diversifiable risk removed" measure, (1-Q)/2, is the same as the limit for Z. However, unlike Q, Z assumes a range of values for any given correlation, corresponding to the actual weighting scheme assumed for a portfolio. The equivalence of Z's maximum and Q's "% diversified" highlights the drawbacks of both measures. We obviously would not set a goal of maximizing Z, since Z is largest when diversification is lowest—that is, when the portfolio's weighting scheme simply plunges into one of the two assets. In these extreme cases, there is no realized diversification at all, regardless of the correlation, so the behavior of the actual portfolio most closely mimics that of the idealized portfolio (the $\rho = -1$ case). While we can adjust our use of Z to recognize this issue (we are not required to assume that we wish to maximize Z), we have no such flexibility for Q, a constant. Q adds no value, even in interpretation, to analysis of the two-asset case.

Figure 3: An Alternative Measure of Diversification Effectiveness



This figure illustrates the behavior of our "Z" measure for relative diversification effectiveness. This metric measures the actual diversification at any weighting scheme relative to the maximum possible diversification (that is, the reduction in portfolio standard deviation at that E(R) between the worst-case scenario of $\rho = +1$ and the best case of $\rho = -1$). Z is maximized at $(1-\rho)/2$ when the weight in either asset approaches 1, and is minimized when $[\sigma_p = 0|\rho = -1]$. Z rises when correlation falls: with lower correlations, the actual reduction in portfolio standard deviation from a given weighting scheme more closely mirrors the maximum possible reduction.

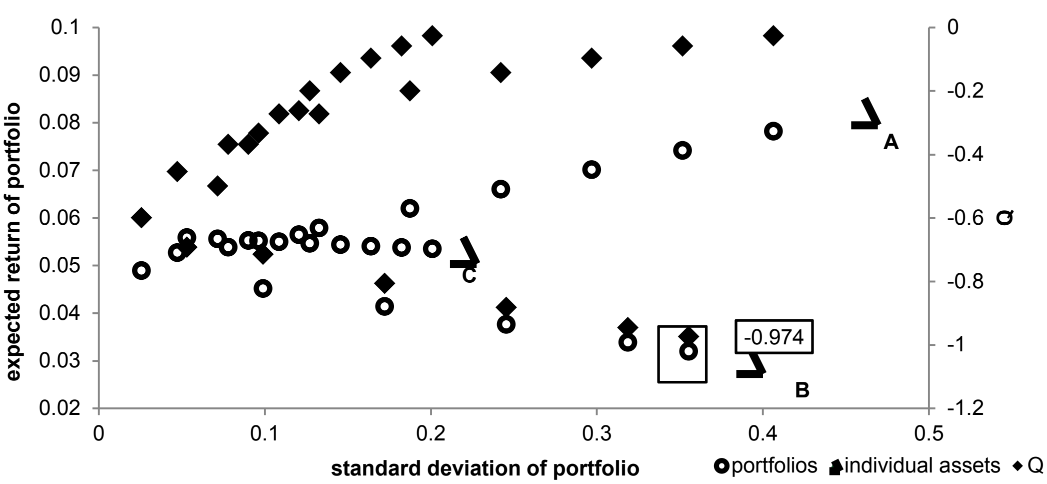
However, Q's adherents would no doubt say that they never intended to use Q is such a simple, unrealistic case. Thus, we turn now to a more general situation: the case of three assets.

<u>Q And Three-Asset Portfolios</u>

One of the claims related to the IPC is that "[t]o eliminate diversifiable risk completely, one needs an intraportfolio correlation of -1" ("Diversification: Finance,"

http://en.academic.ru/dic.nsf/enwiki/2310801). We have already discussed this contention for two-asset portfolios; since Q = ρ in this case, what the statement should assert is that complete diversification is *possible*, given perfect negative correlation, but the correct weighting scheme must be used (i.e., $w_i = \sigma_j/(\sigma_i+\sigma_j)$). In the three-asset (and *n*-asset) case, the interpretation of the assertion is more involved, as we now discuss. (We will ignore the fact that some proponents of IPC insist on applying their diversification assertions to <u>market</u> risk; see, for example, GSphere's discussion. Since Q can lie outside its purported [-1, 1] bounds if short-selling is allowed, and since short-selling is required to "eliminate" market risk [see, for example, Treynor and Black, 1973], we will assume that Q's diversification relates to nonsystematic risk.) Is a Q value of -1 optimal? Let us consider a simple example of three assets, cleverly named A, B, and C. B is perfectly negatively correlated with both A and C (ρ_{AB} = -1 and ρ_{BC} = -1), while A and C are perfectly positively correlated (ρ_{AC} = 1) (Note that, contrary to the apparent claim on some IPC-boosting sites—e.g., Gravity Investments'—not all of the off-diagonal elements can be -1. Beyond the mathematical impossibility, we also have the practical: as Markowitz, 1952, notes, "[t]he returns to securities are too intercorrelated. Diversification cannot eliminate all variance." We consider this further in the next section.) It is impossible to obtain a Q of -1 with these correlations, if we require all three assets' weights to be positive. However, in Figure 4 we plot various portfolios of the three assets (assuming [σ_i, E(R_i)] values of [46%, 8.2%], [39%, 3%], and [22%, 5.3%], respectively) that can be created with positive weights in each. We can come arbitrarily close to Q = -1 by increasing the weight in B. The figure highlights a portfolio with 95% in B and 2.5% in A and C; this portfolio has a Q value of -0.974.

Figure 4: An Example of Q in the Three-Asset Case

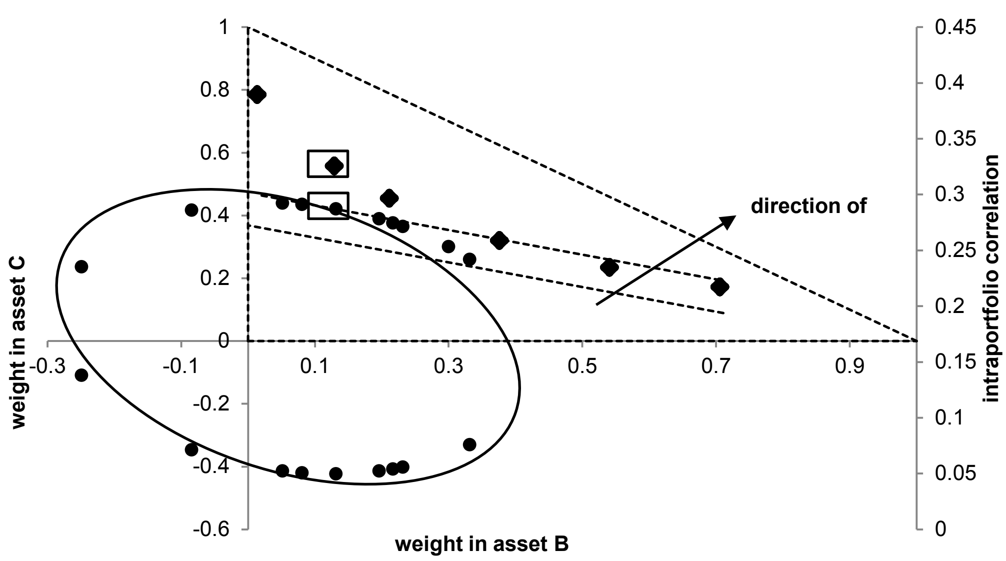


This figure plots various portfolios of assets A, B, and C, along with their corresponding Q values. The highlighted portfolio has the lowest Q—the closest to the purported optimal value of -1—yet is clearly a dominated portfolio.

In the figure, Q values are plotted as circles, using the right-hand axis; portfolios are squares, plotted on the left. The portfolio whose Q is -0.974—supposedly the "best" value of Q—is dominated by almost every other portfolio shown. The Q values for the undominated portfolios tend to rise steadily as expected return rises. We cannot say which of these undominated portfolios is "best," since a choice among them depends on the investor's preferences. (As Markowitz notes: "There is a rate at which the investor can gain expected return by taking on variance, or reduce variance by giving up expected return.") However, we can say that any investor—risk averse by definition—will not prefer the portfolio with the lowest Q. We can also say that that having a Q of (approximately) -1 does not imply that diversifiable risk is eliminated: the highlighted portfolio is so heavily weighted in B that it barely begins to take advantage of its perfect negative relationship with A.

Of course, the situation depicted in Figure 4 is artificial and unrealistic. Since all of the assets are perfectly correlated, our "three asset" case behaves like a series of two-asset cases (as evidenced by the almost linear portfolio curves). In Figure 5, we present a less contrived illustration of a problem with Q. Figure 5 is based on the basic three-asset analysis of Markowitz, as depicted in his Figure 2 (Markowitz, 1952, page 85). (See also Sharpe's 1967 Figure 1, drawn in [E(R_p), β_p] space.) The three underlying assets in Figure 5 are all positively correlated ($\rho_{AB} = \rho_{BC} = 0.2$; $\rho_{AC} = 0.4$). As Markowitz does, we restrict ourselves to nonnegative weights; the dashed triangle that traces the axes from the origin to (0, 1) and (1,0), and whose hypotenuse stretches between those two points, outlines the set of possible weighting schemes. Markowitz shows that the tangency points between isovariance curves (ellipses in [w_i, w_j] space that link portfolios with equal variances) and isomean lines (lines linking portfolios with equal expected returns) trace out the efficient set of portfolios. This set starts at the point of minimum possible portfolio variance, then moves linearly in the direction of increasing expected return until it hits the boundary of the opportunity set (in his case, the hypotenuse of the weighting triangle). (Thereafter, the set is either coincident with an axis or with the boundary.) Investors, being risk-averse return lovers, wish to be on the efficient set.

Figure 5: Q in Markowitz's Isovariance Ellipse/Isomean Space



This figure is based on Markowitz's (1952) Figure 2. The dashed triangle outlines the possible weighting schemes; it assumes no short-selling. The ellipse, outlined with dots, is one of a series of concentric isovariance ellipses. The lines are isomean lines; expected return rises as these lines move to the northeast. Investors choose points on the efficient set, which is determined by the tangencies between isovariance curves and isomean lines. One such point is highlighted, at (.13, .42). However, the Q value for this efficient portfolio (boxed diamond) is not the lowest Q value possible for this level of expected return. Minimizing Q cannot therefore be an appropriate decision rule for a risk-averse investor. (E(R_A) = 45.5%, σ_A = 11%; E(R_B) = 50.3%, σ_B =281%; E(R_C) = 57.7%, σ_C = 25%.)

In Figure 5, we plot one isovariance ellipse, stretching across all four quadrants and outlined by dots. The isomean lines are downward-sloping, given our assets' expected returns, with higher E(R) values to the northeast. One tangency is highlighted, at approximately (.13, .42). The linear efficient set (not pictured) would start at the minimum variance portfolio (also not pictured, but which lies at the center of a system of concentric isovariance ellipses, one of which we have) and pass through this tangency point. According to Markowitz, investors would choose portfolios on this line, within the nonnegative-weight boundaries, according to their preferences. However, this is not what we would expect, given the Q criterion; instead, investors should choose the portfolio with the lowest Q, since it will be the most effectively diversified. Figure 5 plots the values of Q (as diamonds) for portfolios on the same isomean as the given tangency portfolio. The Q value for the tangency portfolio is highlighted—this is the Q corresponding to the point on the efficient set. However, it is not the lowest Q from this isomean; we find lower values of Q as we increase the weight in asset B. These lower Q values must correspond to portfolios with higher variances than the tangency portfolio's: the farther a point is from the center of the ellipses, the higher is its level of variance. Investors would not choose such inefficient points, so they would not want to make their decisions by minimizing Q.

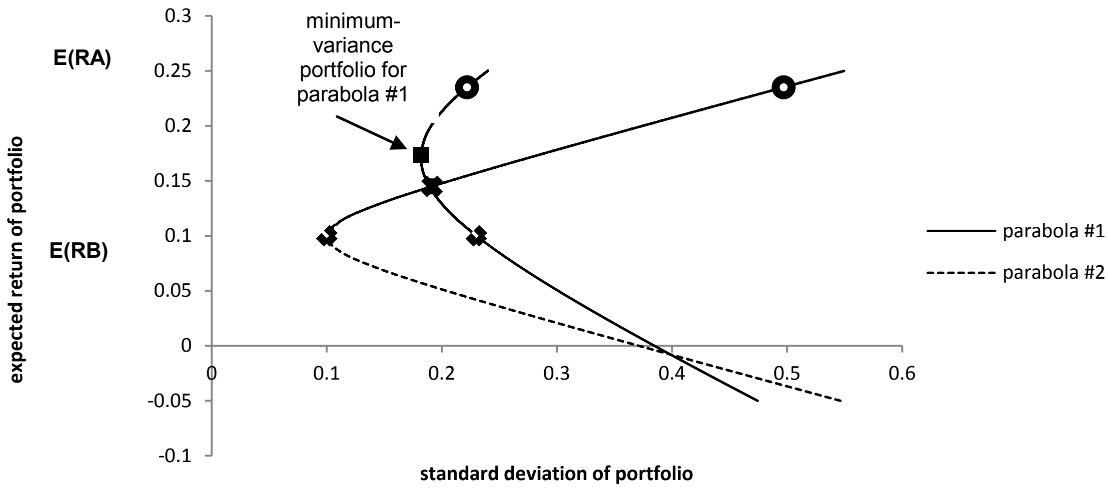
The problem with Q is that it abstracts from the variances of the underlying assets. Figure 6 gives us one more way to think about this problem. In the figure, we show the portfolio parabolas (possible portfolios, plotted in [σ_p, E(R_p) space]) for two sets of assets A and B. The assets A and B used to create the two parabolas have the same expected returns in each case. (Asset A is depicted with circles; asset B by diamonds.) The correlation between the assets is also the same in both cases: 0.2. In neither case is A or B dominated when held in isolation. However, asset A has a higher standard deviation in parabola #2 than in parabola #1, while asset B has a lower standard deviation. Consider the upper point at which the two parabolas cross, which is marked by a star. At this point, both the portfolio on parabola #1 and the portfolio on parabola #2 have the same weighting schemes for A and B, the same expected return, and the same standard deviation. Given the common correlation coefficient, these two portfolios also have the same value for intraportfolio correlation, Q. However, an investor clearly would not be indifferent between the two portfolios—despite the common Q, E(R_p), and σ_p—since the portfolio, while efficient on parabola #2, *is dominated* on parabola #1. Again, these parabolas were constructed using assets differing only on standard deviation, the information Q ignores; they share the same weighting scheme, ρ, and expected returns. Intraportfolio correlation cannot distinguish between them since it abstracts from critical information concerning their relative risk. Yet Q purports to be a risk measure!

Having discussed several problems with Q, we now consider how to incorporate its study into investments courses.

Applications Of Q To Investments Courses

Since portfolio theory and the efficient set are fundamental concepts in investments courses, instructors can find many opportunities to evaluate a portfolio metric like intraportfolio correlation. In this section, we point out just a few of the more obvious examples. We begin with opportunities to augment theoretical discussions, then consider the empirical.The most straightforward theoretical application relates to the discussion of the Capital Asset Pricing Model. After introducing the Capital Market Line (CML), the instructor could ask students to consider the intraportfolio correlation for a portfolio of n assets lying on the Capital Market Line. Since all assets on the CML are perfectly positively correlated, the Q value would be 1, implying that there was no diversification whatsoever. However, since all assets on the CML are already perfectly diversified—they have no systematic risk, by definition—this is clearly a nonsensical implication.

Figure 6: Q and Dominance in the Two-Asset Case



This figure shows two portfolio curves made up of two assets A and B. For both curves, $E(R_A)$ *= 25% and* $E(R_B)$ *= 10%. In parabola #1, the assets' standard deviations are 24% and 23%, respectively; in parabola #2, they are 55% and 10%. Consider the crossover point marked by a star: this portfolio has the same Q on both parabolas, but is dominated on parabola #1. Q cannot distinguish between acceptable and unacceptable (dominated) cases.*

A second theoretical link occurs when considering naïve diversification. Studying the portfolio variance effects of setting all weights to (1/*n*), where *n* is the number of included securities, gives students an early appreciation for the relative inconsequence of individual asset variances and the importance of covariances. Bodie, Kane, and Marcus (2011) provide an example in which they assume that all correlations and variances are the same (var(i) = σ^2 for all i; corr(i,j) = ρ, $i \neq j$). In this case, portfolio variance equals:

$$\sigma_p^2 \quad = \quad \frac{1}{n}\sigma^2 + \frac{(n-1)}{n}\rho\sigma^2 \tag{11}$$

(see their equation 7-21). As *n* increases, it is easy to see that the first term—the contribution of the individual asset variances—goes to zero, while the second term approaches $\rho\sigma^2$. Portfolio variance therefore approaches the common covariance. This is a simple algebraic way to appreciate diversification, and is a nice companion to examination of the covariance matrix.However, what would we learn from Q, given Bodie, Kane, and Marcus's assumptions? Since all pairs of assets have the same correlation, Q becomes simply ρ, as it does for the two-asset case (see equations (3) and (7)). Again, the measure degenerates into something we already knew, and we miss the point of the example—that even naïve diversification diminishes the influence of unique risk.

Moving beyond these theoretical evaluations of Q, investments professors may also introduce IPC in more applied topic areas. We will now consider incorporating Q into a basic course project, into simulations, and into discussions of the Arbitrage Pricing Theory (APT).

As noted earlier, it is common for investments professors to incorporate spreadsheet projects based on real data. Kish and Hogan (2009) stress the need for such a project to allow students to link theory and practice. They suggest having students assume the role of consultants, going through a qualitative and quantitative exercise that culminates with a portfolio recommendation. The focus throughout their project is diversification, so an analysis of Q would fit seamlessly. For example, Kish and Hogan ask students to answer questions like "What insight did you obtain from the correlation matrix," as well as a whole series on estimating the mean-variance efficient portfolios. Students could extend their answers to these sorts of questions by evaluating Q. (For example, when estimating with a scatterplot, as in Kish and Hogan's question #5, students could utilize a plot like our Figure 5.) Kish and Hogan assert that their project embodies a cognitive learning strategy; introducing the analysis and interpretation of intraportfolio correlation would enhance the "elaboration" phase of that strategy.

Other authors' projects offer further opportunities for Q. For example, Ammar, Kim, and Wright's (2008) project incorporates a simulation exercise focused on correlation. Using an equally weighted two-stock portfolio, they demonstrate the power of low correlations to reduce portfolio risk. They suggest—but do not perform—extensions, such as allowing rebalancing. This type of extension could provide an opportunity for students to explore intraportfolio correlation. While Q would not be a sufficient treatment, it would at least open the conversation about the importance of the weighting scheme, which is currently missing from Ammar, *et al.*'s simulation.

More interesting evaluations of Q are possible when students actually set out to find the efficient set. Since identifying mean-variance efficient portfolios is straightforward with Excel, students who learn the process should be easily convinced that a simplistic metric like Q is unnecessary. Carter, Dare, and Elliott (2002) provide a template for finding mean-variance efficient portfolios using Excel's Solver tool. With Solver, students can generate efficient frontiers for *n*-asset portfolios, which they can then compare to reference portfolios (as the authors do with the equally weighted portfolio). Using a graph like Carter, Dare, and Elliott's Figure 10 (a plot of portfolios in [standard deviation, expected return] space), students can easily compare efficient and dominated portfolios, and their Qs, as we did for random portfolios in Figure 4. Students will see Q adds nothing to their appreciation for portfolio diversification.

Students who wish to go further with Excel can follow Arnold's (2002) approach for finding efficient portfolios using the program's matrix multiplication functions. Arnold advocates for introducing students to efficient set mathematics using linear algebra, both to enhance their appreciation for the meaning of efficiency in the *n*-asset case and to link portfolio theory solutions to regression analysis. (See also Martin, Cox, and MacMinn, 1988, p. 682-687.) Given the generality of the templates he provides, he asserts that, "[p]articularly with the aid of a spreadsheet program, multiple asset portfolios are not beyond the comprehension of undergraduate students." Why then, would they need Q?

Tarrazo (2009) reinforces Arnold's emphasis on the relationship between portfolio optimization and linear regression. He notes that both procedures "optimize a quadratic form to minimize the variance of the estimate" (i.e., either minimum squared errors or minimum portfolio variance), and asserts that the only real difference between the two procedures is that portfolio optimization requires full investment (that the sum of the assets' weights is 1), while regression does not. Regression also makes error terms explicit, unlike portfolio optimization—even though the equivalence of the two methods demonstrates that the "financially feasible" portfolios that result from portfolio optimization will "miss their mark."

If students are familiar with these links between portfolio theory and regression (and they undoubtedly will be familiar with regression, given the standard undergraduate business curriculum), then they will be well prepared to evaluate the following assertion by one promoter of intraportfolio correlation:

> [We] created dimensionality to represent the total diversification of a portfolio. More dimensions = more diversification. Normally, we think of having three dimensions to our world plus time as the fourth dimension. In mathematics, there are no limitations to the dimensionality. For example, the branch of physics investigating string theory has discovered that it takes 13 dimensions to attain harmony among their calculations. Every extra dimension that a portfolio has allows it to perform in a simultaneous and independent direction. A perfectly undiversified portfolio is one-dimensional. Think of a dot on a line. The dot can only move up the

> line or down the line. Now imagine a dot placed in a 5 dimensional space. That dot now has freedom to move up or down along each of the five directions. The direction is goes along one axis (dimension) does not connote anything about how it moves along another axis. The measure is patent pending.

Would anyone familiar with regression be startled to learn that many independent variables (or dimensions) may be used to describe the behavior of a dependent variable? Q can also help students studying the Arbitrage Pricing Theory (APT). The APT expands the number of factors used to describe security returns from the CAPM's single factor (in fact, the CAPM is a special case of the arbitrage pricing model), so that students of the APT are quite familiar with the idea of a *k*-factor model. They also should be able to assess the IPC promoter's apparent assertion that more dimensions are better. The APT requires that *k* be less than the number of securities (although that is not much of a constraint in the real world!), and early empirical tests estimated that the number of factors was about four (see the literature summarized in Martin, Cox, and MacMinn, 1988, Chapter 9). More recent empirical work, however, has led the number of applied factors to "explode," according to Huberman and Wang (2005). Since the APT does not specify the return factors—since they are assumed to be common knowledge—researchers are inserting their judgment into the void. (Of the three methods cited by Huberman and Wang to determine factors, one employs a purely mathematical approach—principal component analysis—while the other two involve variations of the researcher's "primarily us[ing] his intuition to pick factors.") The end result is that there are now so many factors that have been used to explain asset returns that Huberman and Wang do not even attempt to list them all. Their assessment of the state of the art is that "[t]he multiplicity of competing factor models indicates ignorance of the true factor structure and suggests a rich and challenging research agenda." But does it also suggest that mere recognition of the potential of multiple factors—the recognition of the IPC promoters—deserves a patent?

CONCLUDING COMMENTS

Intraportfolio correlation is a relatively new metric being advanced by some practitioners as a valuable measure of portfolio diversification. However, it does not appear to live up to its hype. Markowitz notes that his expected value/variance hypothesis implies the "'right kind' of diversification for the 'right reason'." It is not the *number* of securities in a portfolio that generates efficient diversification—it is the *relationships* among those securities. This is undoubtedly the insight driving the advocates of intraportfolio correlation. However, by focusing only on correlation, the standardized measure of comovement, Q ignores the rest of covariance: the standard deviations.In this paper, we evaluated Q using both two-asset and more general three-asset cases. These cases are part of the basic investments curriculum, so should be easily accessible to undergraduate students. Nonetheless, these cases suffice to demonstrate the problems with Q; students will be able to determine that Q is not able to identify optimal portfolios—which is its job. Neumann (2008) reminds us that "[m]any students expect to come out of an investments class with investing advice." If so, and if students want to be able to bridge theory and practice, then a study of Q, an increasingly popular portfolio metric, is appropriate. Familiarity with Q's failings is the kind of practical investing advice budding investors can really use.

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