ADDING MARKOWITZ AND SHARPE TO PORTFOLIO INVESTMENT PROJECTS

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ABSTRACT

Introductory investments courses revolve around Harry Markowitz's modern portfolio theory and William Sharpe's Capital Asset Pricing Model. Nonetheless, the textbook versions of these seminal contributions tend to obscure their economic insights, focusing instead on their mathematical consequences. In this paper, we suggest simple additions to the basic portfolio spreadsheet project that will distinguish the economics (e.g., the market portfolio is efficient) from its necessary consequences (e.g., the beta-expected return relationship is linear). We also show that it is important to use Excel's MMULT function, not Solver, to find efficient portfolios.

JEL: G10, G11

KEYWORDS: Portfolio Theory, CAPM, Investments Pedagogy

INTRODUCTION

An investments course without Harry Markowitz and William Sharpe is unthinkable. Markowitz's (1952) modern portfolio theory and Sharpe's (1964) Capital Asset Pricing Model (CAPM) are the bedrock upon which investments courses are built. Nonetheless, a student does not actually see the concepts as the authors originally presented them; unless she takes one of the few doctoral-level classical papers courses, she sees instead only a textbook distillation. This distillation obscures important detail—in particular, it muddles the distinctions between the authors' (Nobel-prize-winning) economic insights and their purely mathematical consequences.

In this paper, we show how a few simple spreadsheet-based tweaks to a traditional investments project can highlight these critical distinctions. Excel's matrix multiplication functions allow students to identify mean-variance efficient portfolios easily. With those portfolios in hand, they can replicate Markowitz's graphs that show efficient portfolios lying on a line (not on a parabola!), and they can verify Roll's (1977) critique of tests of the CAPM—the observation that it is always possible to derive a linear beta/return relationship ex post. Students who have worked through these sorts of exercises will have a much deeper understanding of modern portfolio theory.

The paper proceeds as follows. After reviewing the relevant literature in the next section, we show how students can recreate Markowitz's presentation of efficient portfolios as the tangencies between isomean lines and isovariance ellipses. Next, we turn to Sharpe's CAPM, demonstrating a simple exercise that will lead students to a linear "Security Market Line." We then briefly summarize and conclude.

LITERATURE REVIEW

This paper links two types of finance literature: the seminal theoretical works of Markowitz (1952) and Sharpe (1964) (now more often referenced than read) and the ongoing pedagogical work on teaching their theories using spreadsheets.

"The Capital Asset Pricing Model…is the centerpiece of modern financial economics," for which "Harry Markowitz laid down the foundation" in 1952 (Bodie, Kane, and Marcus, 1993). Given the centrality of Markowitz's modern portfolio theory and Sharpe's CAPM, it is not surprising that significant periods of investments courses, and numerous chapters of investments texts, revolve around these concepts. For example, Fama and Miller (1972) devote a chapter each to Markowitz's ("pioneering") and Sharpe's concepts—almost one-third of their text, which covers both corporate and investments topics. Bodie, Kane, and Marcus's second edition (1993) devotes two sections (eight chapters) to portfolio theory and capital market equilibrium, all of which are still present in the ninth edition (2011). The 2012 Chartered Financial Analyst curriculum devotes five of 69 readings at Level I to portfolio theory and capital market equilibrium—a significant allocation, given that the scope includes financial accounting, micro- and macroeconomics, ethics, probability, quantitative methods, fixed-income and equity analysis, derivatives, and alternative investments.

However, a standard investments curriculum offers only a highly distilled version of what Markowitz and Sharpe presented in 1952 and 1964. Since the students who do the best in finance are those who really understand the basics (Dubofsky, personal communication, 1988), serious students are well served by considering the sources of their textbook summaries. Nonetheless, that source material is routinely ignored.

It may be that investments professors have tended to shy away from the source material because Markowitz's and Sharpe's treatments require more math than professors—especially of undergraduate students—are willing to assume or able to incorporate into their courses. However, now that Excel and other readily available programs incorporate matrix multiplication functions and optimizers, it is easy to create straightforward exercises that demonstrate portfolio mathematics. These can be seamlessly integrated into the spreadsheet-based projects that are ubiquitous in investments courses already.

There are many examples of investments projects incorporating spreadsheets. Kalra and Weber (2004) outline a basic task-based investments project covering the standard metrics for a single stock; Kish and Hogan (2009) expand this to multiple assets. Neumann (2008) incorporates efficient markets arguments by linking his project to the *Wall Street Journal's* long-running dartboard contest. Moving still further into the "real world," Girard, Pondillo, and Proctor (2005) describe a project incorporating performance attribution analysis. The most recent papers have demonstrated Monte Carlo simulations; for example, Ammar, Kim, and Wright (2008) demonstrate simulations using both Excel's built-in functions and Crystal Ball add-ins.

The papers most relevant for our work are those that show students how to use Excel to find mean-variance efficient portfolios. Carter, Dare, and Elliott's (2002) approach is the most straightforward: they demonstrate optimization using Excel's Solver. Solver is an add-in that performs optimization subject to constraints. The tool is easy to use; students learn it quickly and like it. Bodie, Kane, and Marcus (2011) have incorporated a basic Solver-based exercise into their influential textbook, having students derive the efficient set for a 7-asset international portfolio. They suggest first finding the global minimum-variance portfolio (as do Carter, Dare, and Elliott), then finding additional points on the efficient set of risky assets by iteratively changing the required risk premium.

Despite Solver's power, however, its routine is not always able to find a solution; worse, sometimes it appears to, but has not. Johnson and Liu (2005), who extend Carter, Dare, and Elliott's procedure to allow for investable short sales, suggest using Solver iteratively to derive an optimum. This also may not work. Thus, we also suggest introducing students to Arnold's (2002) matrix multiplication (MMULT) method. It seems less user-friendly than Solver, but students can follow easily the process Arnold outlines. In fact, once the spreadsheet is set up, the matrix method is actually easier and faster than Solver. We will follow Arnold's MMULT method in the work that follows.

Once students are able to identify efficient portfolios, they can learn from experience what Markowitz means when he says, "[t]he point of the isomean line at which [variance] takes its least value is the point at which the isomean line is tangent to an isovariance curve." They see for themselves that the locus of these tangencies is a straight line. As for the CAPM, they can learn why Roll asserted in 1977 that there is "practically no possibility" that "a correct and unambiguous test" of the CAPM "can be accomplished in the future." To demonstrate these opportunities, we turn first to Markowitz.

PROJECT ADDITIONS FROM MARKOWITZ

Investment students, including undergraduates, should read Markowitz's original 1952 paper "Portfolio Selection." Then they will see that mean-variance optimization is not the first or most obvious choice for creating portfolios: Markowitz initially considers (but quickly rejects) plunging into the asset with the highest discounted expected return or diversifying across multiple assets offering the highest expected return. Only then does he present his expected-returns/variance-of-returns rule, complete with definitions of expected return, variance, covariance, portfolio return, and portfolio variance. Students will have seen all of this in their texts; however, seeing its initial presentation provides invaluable reinforcement. More interestingly, Markowitz's illustration of the efficient set of risky assets is different from current textbooks'. Not only does he use variance instead of standard deviation as his risk measure, he also puts return on the *x* axis, so that the efficient frontier is in the southeast corner of the graph, sloping up and left. Having students consider the shape of the frontier in a different space deepens their appreciation for efficiency. It also prepares them for what Markowitz does next—and what modern textbooks do not do at all—presenting the efficient set on its "real" axes: the asset weights.

Markowitz's Linear Efficient Set

The problem with the traditional textbook presentation of the efficient set is that it plots portfolio expected return against portfolio standard deviation—that is, it depicts portfolios in (σ_p, $E(R_p)$) space, in which the minimum-variance set traces out a nice, bullet-shaped curve. As nice as the graph looks, this presentation obscures the true drivers of both expected return and variance: the portfolios' asset weights. Variance does not determine expected return; instead, both are determined by the chosen weights. Appreciating this is critical to understanding portfolio theory.

The most important graph for students is Markowitz's Figure 2, in which he illustrates the determination of the efficient set for a portfolio of three assets. However, before considering this, students should work through the simpler two-asset case. This basic case is interesting because all two-asset portfolios are minimum-variance. Since there is only one weighting scheme that will deliver a target $E(R_p)$ (assuming that the assets have different expected returns), that weighting scheme *must* be the lowest-variance way to deliver that expected return. Students should create multiple two-asset portfolios in Excel and plot them for themselves in (σ_p, $E(R_p)$) space—they will be quite gratified to see their portfolios plot out so beautifully on a parabola.

Moving to the *n*-asset case, however, is much more challenging. Given more than two assets, there are multiple ways to generate a target $E(R_p)$, only one of which is minimum-variance. Markowitz discusses only the three-asset case in detail (mentioning the four-asset case in a footnote, and noting that his results extend to *n* assets). He makes the case in his Figure 2. In this figure, he identifies efficient portfolios in ($weight_1$, $weight_2$) space, where $weight_i$ is the proportion of portfolio funds invested in asset *i*. In this space, Markowitz identifies portfolios offering the same level of expected return and the same level of variance. The former he calls "isomeans"; these plot as lines, the slope of which depends upon the expected returns of the various assets. "Isovariances," on the other hand, form a set of concentric ellipses (assuming no pair is perfectly positively correlated), whose center is the global minimum-variance portfolio. Efficient portfolios maximize expected return for any given level of variance; these portfolios are the tangencies between an isovariance ellipse and an isomean. These tangencies trace out a line in ($weight_1$, $weight_2$) space. Students accustomed to seeing efficient portfolios lying on a parabola may be surprised by this.

They may also be surprised at the ease with which they can now find the efficient set of risky assets. Given two points from Markowitz's linear efficient-set relationship, they can generate the others simply as other points on that line. This is a much easier approach to the three-asset case than is a serial application of Solver.

Figure 1 illustrates the steps I ask my students to follow to generate their efficient set. The graphs are based on a three-asset universe, where assets 1, 2, 3 have expected returns of 15%, 25%, and 10%, respectively, with correlations $\rho_{12} = 0.5$, $\rho_{13} = -0.2$, and $\rho_{23} = 0$.

First, two-asset portfolios of assets 1 and 2 are plotted against various three- asset portfolios. (Note that the equally weighted [EW] three-asset portfolio is shown as a triangle.) This is part of a basic investments project: students create portfolio weighting schemes using two and three assets; they determine the portfolios' expected returns, variances, and standard deviations; they then plot their results in the standard (σ_p, $E(R_p)$) space. With this plot, students can see that their two-asset portfolios fall on a perfect parabola: all two-asset portfolios are minimum-variance. However, they also see that their three-asset portfolios do not behave as nicely. Moreover, while some three-asset portfolios dominate some two-asset portfolios, not all do.

Next, we add three efficient three-asset portfolios. By adding these, students learn that the efficient frontier is still bullet-shaped, but that it has moved to the left (in the preferred direction, to lower standard deviations). Markowitz illustrates the same point, using up to four assets, then asserts (in his Figure 6) that the result holds for *n* assets.

Using Excel, students can see this extension for themselves. As noted earlier, both Carter, Dare, and Elliott (2002) and Arnold (2002) describe Excel-based methods for identifying efficient portfolios. Before continuing with Figure 1, we must digress briefly to consider these two methods—since only one of them proved reliable for our purposes.

First, following Carter, Dare, and Elliott (2002), we used Solver to identify two efficient portfolios based on the EW: the efficient portfolio with the same expected return as EW (the "return-matched" portfolio, $E(R_p) = 16.667\% = E(R_{EW})$), and the efficient portfolio with the same standard deviation as EW (the "variance-matched" portfolio, $\sigma_p = 11.667\% = \sigma_{EW}$). We also used Solver to determine the global minimum-variance portfolio ("GMVP"). We then repeated this process using Excel's matrix multiplication "MMULT" function, as in Arnold (2002). In Table 1, we show our numerical results from these two sets of optimizations.

Table 1 shows that Solver and MMULT give us different portfolio weights for each of the three portfolios. Their results match most closely for the return-matched portfolio; they are farthest apart for the global minimum-variance portfolio. Which portfolios are truly efficient?

We can answer this by turning to Markowitz—to the very relationship we ask students to consider. Our point in asking students to find several efficient portfolios is to show them that these portfolios will lie on a line in ($weight_1$, $weight_2$) space. This is Markowitz's linear efficient set. However, for this to work, students must have truly efficient portfolios. At the bottom of Table 1, we resolve the Solver/MMULT differences by working this relationship backwards: we calculate the slope implied by the portfolios generated by both methods. (For example, in the first row, we find the change in the weight of asset #2 from the global minimum-variance portfolio to the return-matched portfolio, divided by the change in the weight of asset #1.) The portfolios found using MMULT lie on a line in weight space; they therefore are efficient. However, portfolios found using Solver do not lie on a line; Solver's results, while close approximations, therefore are not efficient. Given the problems with Solver, students should use MMULT to complete the generation of the efficient set. (Please see appendix for a brief synopsis of Arnold's MMULT instructions. Also, note that we used the GRG Nonlinear engine to solve our problems. This engine is used for smooth, nonlinear problems. Solver also has an LP Simplex engine for linear problems and an Evolutionary engine for nonsmooth problems.)

(Figure 1 also gives a graphical example of our problems with Solver: As highlighted by the points marked "Solver test," when Solver was constrained to find the efficient portfolio whose expected return equaled 25%, its result fell far outside the efficient set. Two additional results illustrate issues with this optimizer. First, for the global minimum-variance portfolio, Solver's solution returns the w_1 value from [δvariance(w_1, w_2)/$\delta w_1 = 0$], setting $w_2 = 0$: [$(\sigma_3^2 - \sigma_{13})/(\sigma_1^2 + \sigma_3^2 - 2\sigma_{13})$]. However, this ignores the dependence of w_2 on w_1. Second, for the return-matched portfolio, while Solver seems to give the lower variance, it actually violates the weight constraint. In two applications of Solver to this problem, we got two slightly different answers; in both cases, Solver's "solutions" gave weights that added to slightly more than 1. [This happened in no other case during our Solver trials.] Given that the whole point of our project is to show students how efficient portfolios behave, even slight deviations destroy the exercise.)

Table 1: Using Excel's Solver and MMULT to Identify Efficient Portfolios

	(a) GLOBAL MIN-VAR		**(b) SAME E(R) AS EW**		**(c) SAME VAR AS EW**	
	Solver	MMULT	Solver	MMULT	Solver	MMULT
$weight_1$	0.8182	0.8469	0.649125	0.649123	0.5066	0.5070
$weight_2$	0.0000	-0.0306	0.228069	0.228070	0.4141	0.4140
$weight_3$	0.1818	0.1837	0.122807	0.122807	0.0793	0.0791
$E(R_p)$	0.1409	0.1378	0.1667	0.1667	0.1874	0.1874
$variance_p$	0.0073	0.0072	0.009400578	0.009400585	0.013612033	0.013612027
	Solver	**MMULT**				
slope (a)-(b)	-1.349	-1.308				
slope (a)-(c)	-1.329	-1.308				
slope (b)-(c)	-1.305	-1.308				

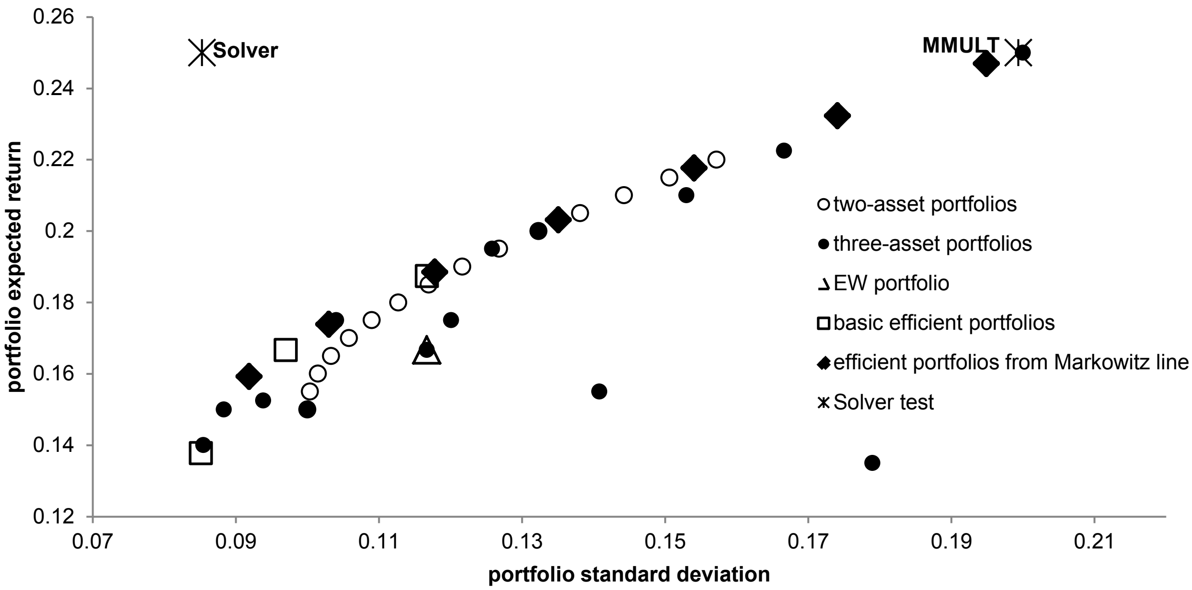
The table duplicates the procedures described in Carter, Dare, and Elliot (2002) ("Solver") and Arnold (2002) ("MMULT"). In panels (a), (b), and (c), respectively, the weight results are shown for the global minimum-variance portfolio, the efficient portfolio whose expected return matches the equally weighted, and the efficient portfolio whose variance matches the equally weighted. The lower panel gives the slope of a line implied by each set of weights in these three panels. Markowitz shows that efficient portfolios lie on a line in ($weight_1$, $weight_2$) space, so any pair of efficient weights will imply the same slope. However, as shown by the varying implied slopes from Solver, its results are not dependable. MMULT's results, however, are: each pair from its three portfolios implies the same slope.

With their efficient portfolios in hand, students can now verify Markowitz's efficient set. Using MMULT, students can quickly generate several efficient portfolios. Plotting those in ($weight_1$, $weight_2$) space they will find that they lie on a line ($R^2 = 1$), just as Markowitz said they would. (We have not illustrated this step in Figure 1, but instructors should be sure to ask their students to do this.) Next, they should use that line to generate additional portfolio weighting schemes. These new portfolios must also be efficient. Finally, as in Figure 1, they should plot these new portfolios in the standard (σ_p, $E(R_p)$) space—where they will fall on a perfect parabola, just as the two-asset portfolios did.

This is the big payoff: making the link between what they know (the parabola) and what Markowitz described (the line). Looking beyond the parabola to Markowitz's line makes students focus on what defines an efficient portfolio of specific assets: the weights chosen. The weights determine expected return and variance; variance does not determine expected return. The optimal weighting scheme for a given level of variance—the one that maximizes expected return—can be identified in ($weight_1$, $weight_2$)

space as the tangency between the given isovariance ellipse and the highest attainable isomean line. This tangency will lie on a line with all of the other efficient portfolios. We have just shown how a simple application of MMULT can help students find that line, and thus to find a simple way to identify other efficient portfolios. Students who wish to explore this further can go on to consider the isovariances and isomeans themselves.

Figure 1: Sequential Derivation of the Efficient Frontier



This figure presents the sequential derivation of the efficient frontier. (We have shown all of the steps in one figure, but instructors may prefer to have their students generate a series of graphs to underscore each step.) First, random two-and three-asset portfolios are plotted (circles). The two-asset portfolios lie on a parabola; the three-asset portfolios are scattered. We then add the global minimum-variance portfolio and the efficient portfolios that match the equally weighted three-asset portfolio on return and variance (open squares). The efficient portfolios found using Markowitz's linear relationship (in weight space) are then added (diamonds) These portfolios lie on a parabola that forms an envelope at the far left of the figure; this is the minimum-variance set. Finally, we add the results from a test of Excel's Solver optimizer and its MMULT function; note that the Solver solution lies outside the minimum-variance set.

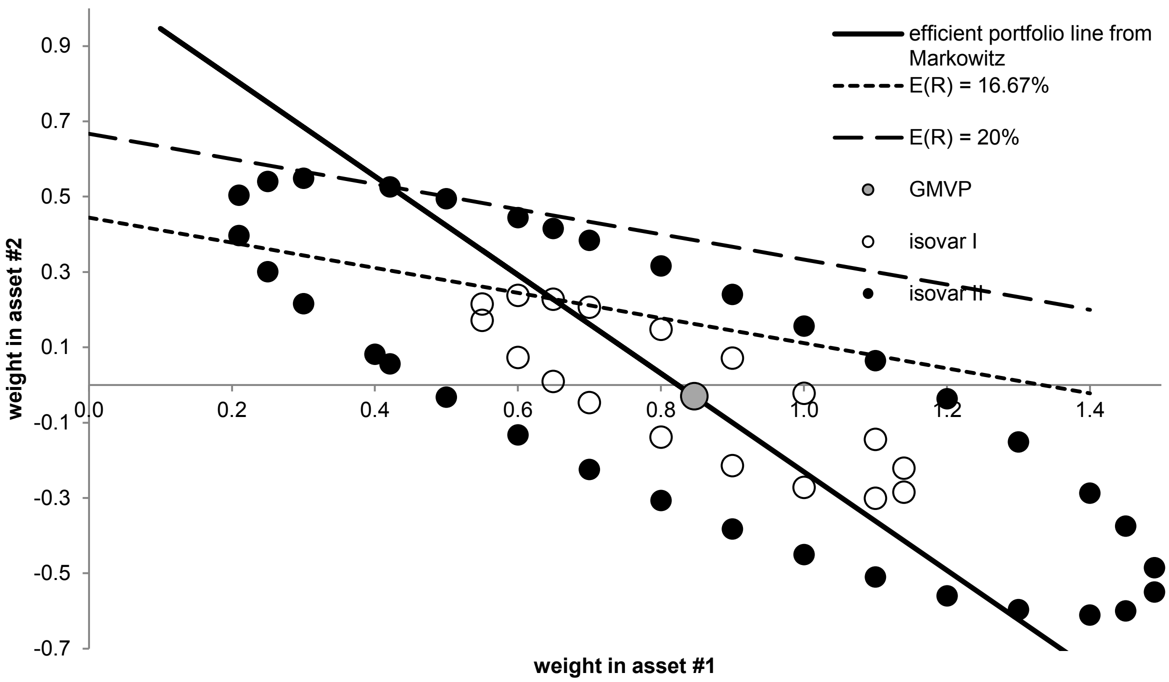
Isomeans and Isovariances

Students of economics are quite accustomed to finding an optimum as the tangency between one curve and another. Adding isomeans and isovariance curves to the linear weight relationship allows students to apply this approach to the portfolio problem and recreate Markowitz's Figure 2. Using Excel, this is not a difficult task.

Our Figure 2 below is an example of this sort of recreation. The red line depicts the weights shown earlier in Figure 1 (c) above. The large dark circle on this line, directly below the *x* axis, represents the weights for the global minimum-variance portfolio. This portfolio forms the center of the series of concentric isovariance ellipses, as we will see below.

Where: $E(R_i)$ is the expected return for asset i, and $E(R_p)$ is the expected return on the portfolio. Isomeans are therefore linear in ($weight_1$, $weight_2$) space. Students will find that isomeans plot as parallel lines, as in our Figure 2. In that plot, the portfolio returns implied by a given isomean get larger as one moves toward the northeast, but this is solely a function of the relative returns of the underlying assets. Students should find it informative to see isomeans that act differently (as in Markowitz's own Figures 2 and 3).

Figure 2: Combining Isomeans and Isoquants to Find Markowitz's Linear Efficient Set



This figure recreates Markowitz's Figure 2. It shows that efficient portfolios are identified as the tangencies between isomean lines and isovariance ellipses. The curve linking all of these tangencies is a line in (weight$_1$, weight$_2$) space.

Each point on the efficient set line is a tangency between an isomean and an isovariance. The isomeans link portfolios with a specified expected return. Markowitz gives the equation for isomeans for portfolios of three assets:

$$\text{weight}_2 = \frac{E(R_p)-E(R_3)}{E(R_2)-E(R_3)} - \frac{E(R_1)-E(R_3)}{E(R_2)-E(R_3)} * weight_1, \qquad (1)$$

The equation for the isovariances—the curves linking portfolios with a specified variance—is more unwieldy, as variance terms are. We use the following simplifications, letting:

$a = \sigma_1^2 - 2\sigma_{13} + \sigma_3^2$

$b = \sigma_2^2 - 2\sigma_{23} + \sigma_3^2$

$c = \sigma_{12} - \sigma_{13} - \sigma_{23} + \sigma_3^2$

$d = \sigma_{13} - \sigma_3^2$

$e = \sigma_{23} - \sigma_3^2$

(so that the variance of the portfolio, V, is $w_1^2(a) + w_2^2(b) + 2w_1w_2(c) + 2w_1(d) + 2w_2(e) + \sigma_3^2$), and

$x = V - w_1^2(a) - 2w_1(d) - \sigma_3^2$

$z = 2w_1(c) + 2(e)$.

Now, a straightforward application of the quadratic formula gives us the weights for asset 2 for a given weight$_1$ and V, the target variance:

$$\text{weight}_2 = \frac{-z \pm \sqrt{z^2 - 4b(-x)}}{2b} \qquad (2)$$

Using (2), students can generate isovariance ellipses easily. First, they can use MMULT to find the global minimum-variance portfolio; this forms the center of the concentric ellipses. The weight_1 value for this portfolio (which we will call w_{1MVP}) is therefore the starting point for their choices of weight_1 for each ellipse they plot. For a given variance, they then should choose a series of weight_1 values around w_{1MVP}. In choosing their weight_1 values, it is sufficient for them to add and subtract the following "radius" value from w_{1MVP}:

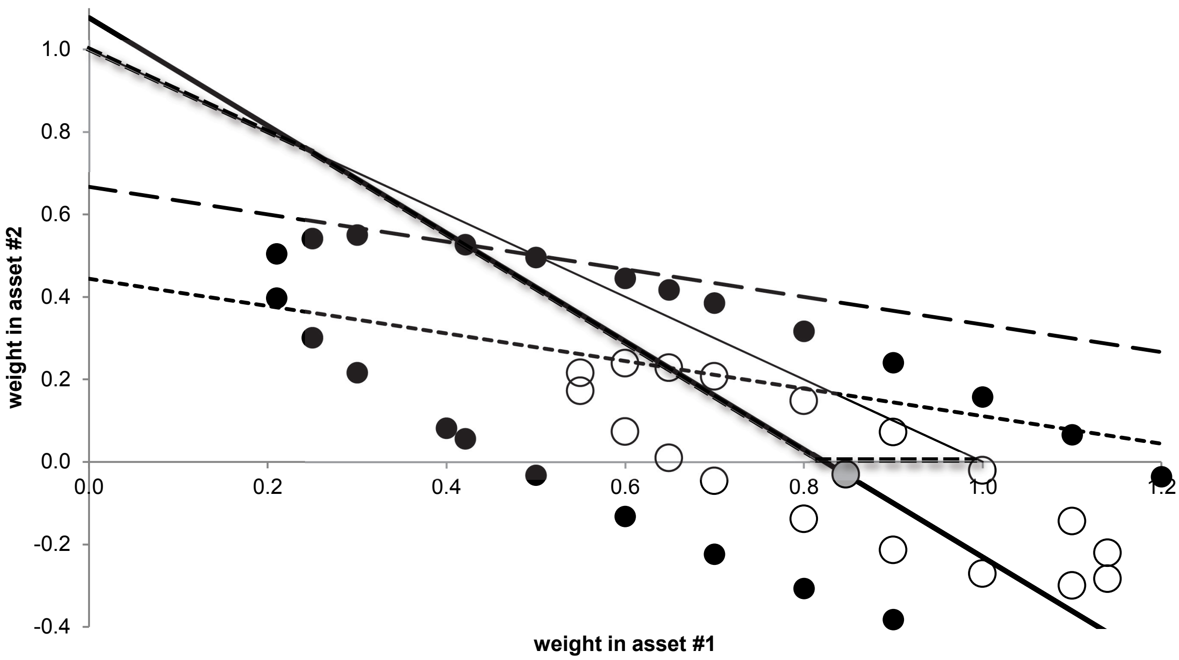
$$\text{radius} = \frac{(V - \sigma_3{}^2) + \frac{(2d)^2}{4a} + \frac{(2e)^2}{4b}}{a}. \qquad (3)$$

(This radius is almost certainly too wide; Excel will therefore return an error message #NUM! for the weight_2 values corresponding to weight_1 values outside the ellipse. Students can just delete those weighting schemes.) Once they have determined the relevant weight_1 domain using (3), they should solve equation (2) twice for each relevant weight_1: once by adding the radical term in the numerator, and once by subtracting it. This will give them their weight_2 values, and they are ready to plot.

Our Figure 2 plots two isovariance ellipses. "Isovar I" shows portfolios whose variance equals that of the "variance-matched" portfolio ($\sigma_p = \sigma_{EW}$). This isovariance ellipse is tangent to the 16.67% isomean exactly where is should be: at the point previously identified by Markowitz's efficient-set line. Similarly, "isovar II" (portfolios with the same variance as the efficient portfolio whose E(R) = 25%) is tangent to the 25% isomean at the point where that isomean intersects on the efficient-set line. Now students can see that Markowitz's efficient-set line is simply marking the tangencies of the isovariances and isomeans. Microeconomics students have seen relationships like this before (for example, an individual's demand curve simply marks the points of tangency—albeit on different axes—between her indifference curves and her income constraint, as the latter rotates around its *y* intercept to illustrate new prices for the *x* good). Plotting these tangencies reinforces students' appreciation for efficiency.

They can then take one last step. Markowitz did not allow short sales. Since he required nonnegative weights, all acceptable portfolio weighting schemes had to fall within a triangular boundary: from the origin to (1,0) and (0,1) along the *x* and *y* axes, respectively, with a hypotenuse from (1, 0) to (0,1). (Portfolios identified by points along the boundary assign a weight of 0 to one of the three assets.) This boundary is marked by a bold black line in Figure 3, which also reproduces the curves from Figure 2. Note that the efficient-set line lies inside the triangular boundary for the most part, but does extend outside it. Thus, just as Markowitz did in his Figures 2 and 3, we must depart from efficiency when the efficient outcomes lie outside the allowed boundary. In our example, the resulting revised efficient set is kinked twice: it follows the hypotenuse from (0, 1) until it meets the true efficient set line; it then follows the efficient set line down to the *x* axis, after which it track the *x* axis to (1, 0). Reevaluating their efficient sets in light of Markowitz's short-selling constraints helps students appreciate the techniques of Johnson and Liu (2005) on their portfolio project. It also clarifies the assumptions underlying Sharpe's Capital Asset Pricing Model, to which we turn next.

Figure 3: Prohibiting Short Sales Creates a Kinked Efficient Set



In this figure, we trace out the efficient set, given Markowitz's short-selling constraints. Portfolio weights must lie within the triangle defined by the x and y axes and the line from (0,1) to (1,0). The efficient set is therefore kinked in our example. It first follows the hypotenuse of the boundary triangle from the y axis at (0,1) until it hits the efficient set; it then follows the efficient set down to the x axis, moving through the triangle of allowable weights; it then follows the x axis to the point (1,0).

PROJECT ADDITIONS FROM SHARPE

Students often leave their basic finance courses thinking that the Capital Asset Pricing Model (CAPM) *is* the equation for the Security Market Line: $E(R_i) = r_f + \beta_i*[E(R_M) - r_f]$ (where β_i is the systematic risk of asset i, r_f is the return on the riskless asset, and $E(R_M)$ is the expected return on the market benchmark). However—given unlimited borrowing or lending at a risk-free rate, homogeneous expectations, and no market frictions, among other assumptions—the economic insight of the CAPM is that the market portfolio is efficient. If the benchmark is efficient, the relationship between beta and expected return *must* be linear. "…[W]e may arbitrarily select *any* one of the efficient combinations, then measure the predicted responsiveness of *every* asset's rate of return to that of the combination selected; and these coefficients will be related to the expected rates of return of the assets in exactly the manner pictured"—a line (Sharpe, 1964; emphasis original). Roll (1977), in his famous critique of tests of the CAPM, puts it this way:

> There is an 'if and only if' relation between return/beta linearity and market portfolio mean-variance efficiency…In *any* sample of observations on individual returns…there will always be an infinite number of ex-post mean-variance efficient portfolios. For each one, the sample 'betas' calculated between it and individual assets will be exactly linearly related to the individual sample mean returns. In other words, if the betas are calculated against such a portfolio, they will satisfy the linearity relation *exactly* whether or not the true market portfolio is mean-variance efficient. (emphasis original)

Using Excel, students can easily prove that beta/return linearity is a consequence of the efficiency of the "market." We present a simple exercise in Table 2. We start with price data for three assets, Harley

Davidson stock (HOG), Ford stock (F), and the S&P 500 market index. We then found the daily returns, averages, sample standard deviations, and covariances. Doing these sorts of calculations is standard for basic investments spreadsheet projects.

Given the covariances, we used Excel's MMULT function to solve for the efficient portfolio that has the same expected return as the equally weighted portfolio (a "return-matched" portfolio, where $E(R_p)$ = 0.3591% in this example). These weights were -.0822, -.2492, and 1.3314 for HOG, F, and S&P, respectively. We used these weights to create portfolio P, as shown in Table 2. Finally, we calculated "betas" using both the S&P500 and efficient portfolio P as benchmarks (so, for example, the "beta with S&P" for HOG is cov(HOG, S&P)/var(S&P)). We repeated this process using Solver to identify the efficient portfolio (w_{HOG} = .0956, w_F = 0, $w_{S\&P}$ = .9044). We plot the results in Figure 4.

The beta/return relationships based on the S&P500 and Solver's "efficient" portfolio are decidedly nonlinear. Early empirical results like this seemed inconsistent with the CAPM. However, as Roll argued, the CAPM is an ex ante model, and testing the beta/return relationship using an arbitrary benchmark using ex post data could not be expected to result in the linearity implied by the theory. On the other hand, if it is linearity that we want, we can always find it—as long as we are careful to choose a benchmark that we know if efficient. Since P was efficient, using it as the benchmark did lead to a linear beta/return relationship—a perfect one, just as is implied by the mathematics of "efficiency."

Table 2: Example of a Simple "Security Market Line" Exercise

Weight in Efficient Portfolio, P:	**-0.0822**		**-0.2492**		**1.331**		
	HOG		**F**		**S&P**		**P**
DATE	**price**	**return**	**price**	**return**	**price**	**return**	**return**
16-Mar-12	$49.39	0.0012	$12.51	-0.0302	1,404	0.0011	0.0089
15-Mar-12	$49.33	-0.0086	$12.90	0.0016	1,403	0.0060	0.0083
14-Mar-12	$49.76	0.0081	$12.88	0.0142	1,394	-0.0012	-0.0058
13-Mar-12	$49.36	0.0260	$12.70	0.0217	1,396	0.0181	0.0166
12-Mar-12	$48.11	0.0262	$12.43	-0.0119	1,371	0.0002	0.0010
9-Mar-12	$46.88	0.0198	$12.58	0.0096	1,371	0.0036	0.0008
8-Mar-12	$45.97	0.0081	$12.46	0.0180	1,366	0.0098	0.0079
7-Mar-12	$45.60	0.0106	$12.24	0.0124	1,353	0.0069	0.0052
6-Mar-12	$45.12	-0.0293	$12.09	-0.0297	1,343	-0.0154	-0.0107
5-Mar-12	$46.48		$12.46		1,364		
Mean:		0.0069		0.0006		0.0032	0.0036
Sample Standard Deviation:		0.0177		0.0199		0.0091	0.0083
Beta With S&P:		1.334		1.630		1.000	0.815
Beta With Efficient Portfolio, P:		1.053		0.953		0.994	1.000

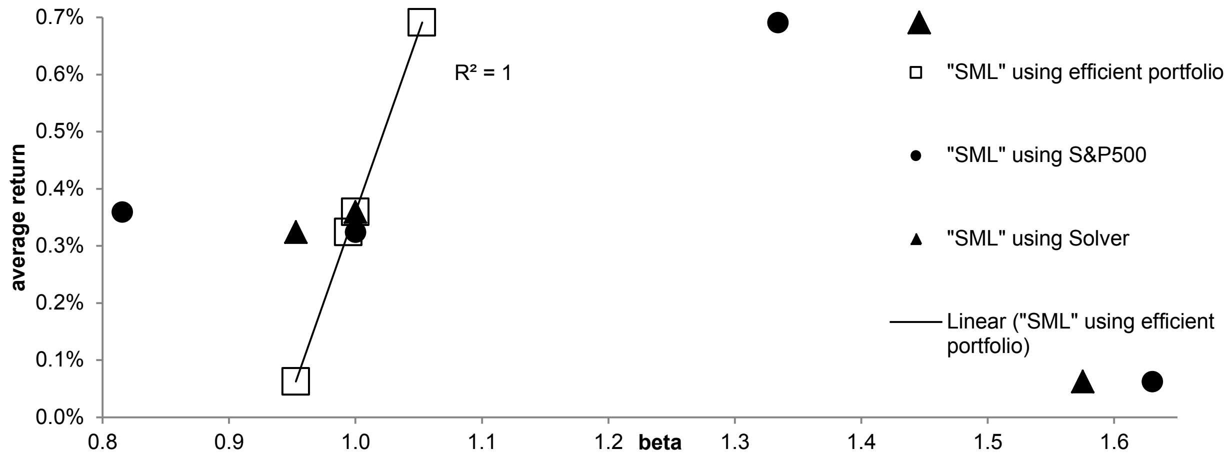
	HOG	**F**	**S&P**	**P**
HOG	0.0003			
F	0.0002	0.0004		
S&P	0.0001	0.0001	0.0001	
P	0.0001	0.0001	0.0001	0.0001

The data in Table 2 were used to create the beta/return relationship graphed in Figure 4. Price data is from Yahoo! Finance. The individual assets' summary statistics are listed below their daily returns; covariances are presented in the matrix at the bottom of the table.

CONCLUDING COMMENTS

Using a spreadsheet project in investments courses has become almost a given. However, while these projects harness the power of Excel to perform basic calculations on real data, they have not yet exploited it to explore the theoretical underpinnings of portfolio theory. In this paper, we show several straightforward investments applications of Excel's matrix multiplication functions (following Arnold, 2002). Students will not only learn a valuable Excel tool (and the potential problems with its Solver optimizer), but will also become much more familiar with portfolio mathematics.

Figure 4: The Ex Post "Efficient Set"



This figure demonstrates the equivalence between an efficient benchmark and a linear return/beta relationship. Using ex post data, inefficient benchmarks (here, both the S&P500 and the portfolio identified by Solver) give return/beta relationships that are nonlinear. However, an efficient benchmark—always available ex post—will lead to a linear relationship, as demonstrated by the trend line.

Markowitz's (1952) figures look daunting, but students—once they get past their fear—find them extremely helpful. Being able to recreate the graphs in Excel demystifies them. Using MMULT allows students to identify efficient portfolios, which is more than half the battle. Given a few efficient weighting schemes, students can verify that the optimal portfolios—the tangencies between the isomean lines and the isovariances ellipses—lie along a line in ($weight_1$, $weight_2$) space. This is surprising for students, who are accustomed to the traditional parabolic representation. Using the linear relationship they find, they can work back to that parabola by finding more efficient portfolios from the line, then replotting them in (σ_p, $E(R_p)$) space. This is the big payoff: linking Markowitz's line directly to the traditional parabola.

We can also use Excel to work through some issues with the Capital Asset Pricing Model. Most importantly, we can demonstrate part of Roll's critique of the CAPM: that the linearity of the beta/return relationship does not prove Sharpe's theory, but is simply a mathematical consequence of the efficiency of the benchmark portfolio. Students can prove this by using MMULT to identify an ex-post efficient portfolio from data they choose on their own, then using that portfolio as the benchmark for beta calculations. Their beta/return relationship *will* be linear—just as Roll said. Students will then be in a much better position to distinguish the ex-ante nature of the economic contribution of the CAPM—that the market portfolio is efficient—from the ex post consequence of efficiency, and they will be better able to evaluate the empirical tests of the CAPM. They will also become immune from the error of equating the SML equation with the CAPM theory.

Markowitz's and Sharpe's theories involve abstruse and complex applications of portfolio mathematics—much more technical knowledge than a professor would expect in an introductory investments class. Traditional textbooks therefore gloss over this background, and present the theories in an easily digestible way. However, it is no longer necessary to avoid the foundations. Using Excel, students can use efficient portfolios to guide their study of the Modern Portfolio Theory and the Capital Asset Pricing Model, without having to understand quadratic programming. Adding this explicit consideration of the consequences of efficiency—with a simple application of MMULT—will enhance significantly students' appreciation for investments.

APPENDIX: USING MMULT

Arnold (2002) describes a relatively easy way to generate efficient portfolios using Excel's matrix multiplication functions. To find the global minimum-variance portfolio, the expression is:

=MMULT(MINVERSE(MINCOV),L)

where MINCOV is Arnold's name for an (n+1)x(n+1) matrix created by surrounding the covariance matrix with a row and column comprised of 1s (except for a 0 in the bottom right cell), and L is an (n+1)x1 vector with 0s in very cell but the bottom one (which has a 1). Once the two input matrices are created, follow Arnold's three-step process:

1. Highlight (n+1) cells in a column (these cells will hold the n minimum-variance weights in its top n rows; the bottom cell holds the Lagrange multiplier term).
2. Type in the equation above, entering the matrix ranges for MINCOV and L.
3. Simultaneously hit Control-Shift-Enter.

Beware: Failing to highlight (n+1) cells (just typing into one cell) and/or hitting only Enter will result in only the weight for the first asset.

The global minimum-variance weights for the example in the current paper were found as follows:

MINCOV				L	weights
0.01	0.01	-0.005	1	0	**0.84694**
0.01	0.04	0	1	0	**-0.03061**
-0.005	0	0.0625	1	0	**0.18367**
1	1	1	0	1	-0.0072

To find the efficient weights for a portfolio with a specified expected return, add another row and column to the MINCOV matrix (which will now be (n+2)x(n+2)); the first n cells on the new row, and the first n rows of the new column, contain the portfolio assets' expected returns. (The remaining cells hold 0s.) Arnold calls this matrix EFFCOV. To the vector L, add a new cell at the bottom, containing the target expected return; call this K. Now, proceed as before, but highlight (n+2) cells for the result vector. The desired weights will again be in the top n rows of this vector. The example below illustrates the process for the efficient portfolio that has the same expected return as the equally weighted portfolio (16.67%).

Creating the EFFCOV/K matrices makes generating efficient portfolios extremely easy: simply changing the target expected return—the bottom cell in the K matrix—automatically generates the new weights. This is much easier than the iteration process using Solver that Bodie, Kane, and Marcus (2011) advocate.

EFFCOV					K	weights
0.01	0.01	-0.005	1	0.15	0	**0.64912**
0.01	0.04	0	1	0.25	0	**0.22807**
-0.005	0	0.0625	1	0.10	0	**0.12281**
1	1	1	0	0	1	0.00303
0.15	0.25	0.10	0	0	0.1667	-0.0746

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