A MARKETING COMPETITION WITH A FINITE TERMINATION TIME: SOME DIFFERENTIAL GAMES

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ABSTRACT

This paper attempts to develop some theoretical insights into the dynamics of marketing campaigns. It studies a model where two firms are competing in an advertising campaign with sales at a specified termination date and asks how the trajectory of resource expenditures would change over time. Among its main findings are that the dynamics of competition force the firms to accelerate their expenditure on advertising throughout the entire time period.

INTRODUCTION

Marketing managers are fully aware that competition over market shares may involve long term strategic considerations, but so far the literature in the field only contains a small number of abstract analyses of these dynamics. The result is that, although we have a wealth of detailed information on particular campaigns, we have little by way of theoretical insights into the factors that shape the development of campaigns over time. This paper is a contribution to redressing this balance.

The paper will consider the following, fairly straightforward, question. Imagine we have two firms engaged in a competition to market a product over time in order to get the greatest payoff, measured in terms of customer support or sales, at some terminal time. This might happen, for example, where rival firms are launching a new product and attempting to build public support in anticipation. The most obvious examples would be pre-release advertising for a film, or motor car companies attempting to launch a new vehicle at a specified date. Other examples would be the rivalry between Boeing and Airbus prior to the aircraft being available, or two companies proposing rival developments of large retail or housing complexes and undertaking preliminary marketing in order to build support prior to the sale date. Another, not so obvious, example that has largely been ignored in the marketing literature is competition between political parties with the terminal time being the election date.

The question I want to consider is, what is the optimal trajectory of advertising expenditure over time if the only assumption made about individuals is that the rate of change in their support at any given time will increase if advertising expenditure is increased? In other words it is not assumed that consumers are forgetful, or that they accumulate memories.

To give the study a concrete point of reference, it often seems to be the case that advertising accelerates towards the terminal point of the sort of campaign mentioned. Although not too much can be made of this sort of casual empiricism it does open some questions about how this sort of acceleration can be explained. Clearly it would make sense if it were simply assumed that customers forget previous messages. Could it be explained in the absence of such assumptions?

The main finding is roughly that, for the model studied, the firms mostly accelerate their expenditure for the entire period up to the terminal time in a way consistent with the previous observation. I demonstrate this for an open loop and closed loop game and also compare these results with a leader follower game, although is not possible to produce analytical solutions for every instance.

What these findings tell us is that any acceleration in marketing expenditure can be explained as the result of the process of competition and does not require assumptions about customers being forgetful or

responding to the most recent messages. To get some feel for the meaning of this consider the case of a single firm. If customers were not forgetful, there would be no reason to increase resource expenditure as the deadline approached. It may as well spend all its resources at the beginning of the time period. If they are forgetful all resources should be spent at the end. The fact that the firms in competition do neither of these tells us something about the dynamics of the competitive process itself.

It will be noted that most of the results of the paper could not have been guessed in advance. Take the simple case, analyzed below, where the fraction of market share for each firm does not alter over the entire period to the end of the campaign. Why, for example, should firms keep accelerating their expenditure on marketing in order to stand still? This is often known as the Red Queen effect after the Red Queen's comment to Alice that it is necessary to run faster to stand still (Carroll, 1960).

This is not the first attempt to study the dynamics of marketing in management science. The dynamics of advertising for a single firm has been extensively studied since the Nerlove-Arrow (1962), and Vidale-Wolfe (1957) models. There have also been studies of advertising competitions between oligopolies in discrete time such as that of Villas-Boas (1993) and Steenkamp (2005) and in continuous time such as that of Fruchter and Kalish (1997) and Chintagunta and Vilcassim (1992). This paper is closest to the study by Fruchter (1997). It differs in that the competition has a finite termination date. In addition I also study a game where one firm decides its expenditure first.

I set out the paper as follows. A model of the dynamics of marketing activities is developed in Section two. In section three, the model is analyzed for the case where both firms formulate their strategies at the beginning of the period. Section four deals with the case where firms adjust their strategies as they get feedback. Section five deals with the case where one firm is a leader and firm two follows.

THE MODEL

Suppose that there are two firms engaged in a competition to market a product at some specified time in the future and they are competing for a fixed pool of customers. This pool is large with each customer having equal purchasing power. Each firm can spend any amount it likes on marketing, although there is a cost associated with this which keeps the expenditure in bounds. It is assumed that the rate of change in the fraction of the potential customers that intend to purchase the product of firm *I* at the release date increases with *i's* expenditure if nothing else changes. The firms might decide on the pattern expenditures in a number of ways.

1. They might make a plan at the start of the game about how much to spend at each instant and hold to this plan for the entire period until the campaign ends. This is called an open loop marketing game.

2. They might change their strategies at each instant in response to information on the levels of support. This is called a closed loop game.

3. One firm might delay its actions until the other has formulated its campaign. In this case firm one is the leader and decides on it expenditure of resources knowing that firm two will then choose a pattern of expenditure to get the greatest share of the pool for itself. This is called an open loop game.

The formal specifications are as follows. Subscripts 1 and 2 refer to the firms. Time is written t and the period until the game ends is normalized to [0, 1]. The resource expenditure of firm one at time t is $u_1(t)$ and of firm two is $u_2(t)$ and the cost of expenditure is $\frac{c_1}{2}u_1(t)^2$ and $\frac{c_2}{2}u_1(t)^2$. The fraction of the buyers that is disposed towards firm one's product at time t is written $x(t)$ and it is assumed that the number of

buyers is sufficiently large that x(t) can be approximated with a continuous function. This means that the fraction of the total buyers for firm two is $y = (1 - x)$.

It is assumed that the effect of spending by the firms on the rate of change in x depends on the fraction of support they control and declines as this fraction increases. This could be explained, for example, by assuming that the most easily persuaded potential customers are won over first and that less easily persuaded customers require more effort. This can be written as:

$$\dot{x} = k_1(1 - x)u_1 - k_2 - k_2 x u_2 \tag{1}$$

for k_1 and k_2 constants.

Note that the firms are only interested in the payoff at the terminal time but that the costs of expenditure will be incurred across the entire time period. In order to capture this it is assumed that they attempt to maximize payoff functions of the form

$$J^1 = -\int_0^1 \frac{c_1}{2} u_1^2 dt + x(1) \text{ and } J^2 = -\int_0^1 \frac{c_2}{2} u_2^2 dt + x(1) \tag{2}$$

subject to the dynamics in Equation (1).

What we want to find out is how expenditure on advertising and the trajectory of support will change over time. This is studied in what follows. Details are provided in the appendices.

THE OPEN LOOP ADVERTISING GAME

The optimum program in this case is for each firm to accelerate its expenditure for the entire time period. The way in which the trajectory of support changes along an optimal path depends on a parameter that captures the relative cost of influencing the dynamics. Where the cost of advertising for firm one is sufficiently high relative to that of the firm two its fraction of support is decreasing. If firm one's relative cost is sufficiently low, support increases. In order to show all this, the problem for each firm is solved. Details of the analysis are in Appendix 1. What we get is

$$u_1 = \alpha_1 \frac{k_1}{c_1}(1 - x) \text{ and } u_2 = -\alpha_2 \frac{k_2}{c_2} x \tag{3}$$

$$\dot{\alpha}_1 = (k_1 u_1 + k_2 u_2)\alpha_1 \text{ and } \dot{\alpha}_2 = (k_1 u_1 + k_2 u_2)\alpha_2 \tag{4}$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are called the costate variables for the problem and must satisfy the terminal conditions $\alpha_1(1) = 1$ and $\alpha_2(1) = -1$ for a solution to be optimal. Solving Equation (4) gives $\alpha_1 = \alpha_2$. Writing $\alpha = \alpha_1 = -\alpha_2$ for $I = 1,2$ gives

$$\dot{u}_1 = \frac{k_1}{c_1} \alpha k_2 u_2 > 0 \text{ and } \dot{u}_2 = \frac{k_2}{c_2} \alpha k_1 u_1 > 0 \tag{5}$$

which means that both firms are increasing their expenditure for all time. Taking the second derivative of these functions shows that the rate of increase in expenditure is also increasing. This shows something of the power of the formal analysis as this result would not have been obvious, to me at least, from the set up of the marketing competition.

In order to analyze the way in which the fraction of support changes substitute Equation (5) into Equation (1). This gives

$$\dot{x} = \alpha_1 \left(\frac{k_1}{c_1}(1-x)^2 - \frac{k_2}{c_2}x^2 \right) \tag{6}$$

along the optimal path. It follows that $\dot{x}$ has the same sign as

$$\varphi = 1 - 2x + x^2 \tag{7}$$

where

$$\kappa = \frac{k_2^2}{c_2^2} \bigg/ \frac{k_1^2}{c_1^2}$$

For $x \leq 1$ the positive root for φ is $r = \dfrac{1-\sqrt{\kappa}}{1-\kappa}\, \kappa \neq 1$ with $\dfrac{\partial r}{\partial \kappa} < 0$. It follows that for $x < r$ we have $\dot{x} > 0$ and for $x > r$ we have $\dot{x} < 0$. What this tells us immediately is that, the fraction of the market going to firm one is always either increasing, decreasing or stationary. See Figure 1.

We can get a better mental picture of what is happening if we rewrite κ as $\dfrac{c_1}{k_1^2} \bigg/ \dfrac{c_2}{k_2^2}$ and interpret $\dfrac{c_1}{k_1^2}$ as an index of the cost of impact of firm one's expenditure on advertising. Even though we are working with k_i^2 it will still have the required properties for changes in c_i and k_i. This allows us to interpret κ as the ratio of the cost of impact of firm one's advertising expenditure over the cost of impact of firm two's expenditure. Call this impact cost. This gives three cases to consider.

Figure 1: The Dynamics for x.
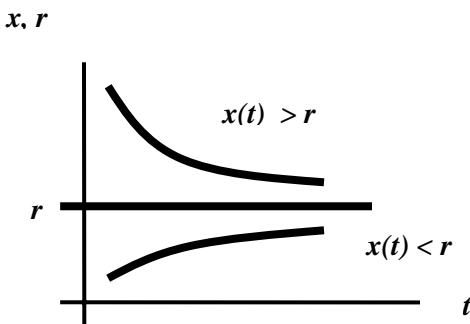
Case 1: Impact Costs Equal

(i). The firms are symmetrical in the sense that each has the same fraction of the total market at the beginning of the competition. It is immediate from Equations (3) and (7) that

$\dot{x} = 0$ and $u_i = u_2$

which means that firms are spending more and more in as time progresses in order to stand still. This is the Red Queen effect mentioned in the introduction.

(ii). The firms are asymmetrical and firm one has more initial support than firm two, perhaps as the result of brand recognition or previous reputation. In this case assume that the coefficients in the dynamic equation are the same and $k_1 = k_2 = \bar{k} > 0$ a constant. This gives

$\dot{x} < 0$

and the fraction of total support for firm one falls for the entire period and that of firm two increases. In addition it can be shown that expenditure for firm two is higher than that for firm one for the entire period. In other words, it pays the firm with the lower level of initial support to try harder for the entire period.

Case 2: Impact Cost of Firm One is Less Than Firm Two

The qualitative results are the same for the case where firms have the same initial support and for the case where initial support is greater for firm one. The fraction of support for firm one increases for all time for κ sufficiently small. For the special case $k_1 = k_2 = \bar{k}$, expenditure by firm two is increasing faster than expenditure by firm one from Equation (5). It also follows immediately from Equation (1) that expenditure for firm one is greater than for firm two for all time.

Case 3: Impact Cost of Firm One is Grater Than Firm Two

In this case support for firm one is decreasing and, if we again set $k_1 = k_2 = \bar{k}$, we get from Equation (3) that $u_1(0) = u_2(0)$. In addition firm one is increasing its expenditure faster than firm two. What seems to be happening is that firm one compensates for its lower impact cost by spending less at the beginning and accelerating its expenditure towards the end of the campaign period.

THE GAME WITH INFORMATION ON SUPPORT AND EQUAL IMPACT COSTS

The closed loop case where firms adjust their advertising expenditure according to information on the level of support at each instant is only analyzed for the parameter values $\dfrac{k_i^2}{c_i} = k$, for $i = 1,2$. The trajectory is essentially the same as the trajectory in the open loop case. When firms are asymmetrical and firm one has more initial support we also have the result that, for $k_1 = k_2 = \bar{k}$, the firm with less initial support spends more for the entire time period. An unexpected feature of this case is that firms spend less on marketing at each instant and hence across the entire time period than in the closed loop game, even thought the end results are the same. It is not clear why this is the case. One explanation might be that, if firms are able to adjust their strategies at each instant they must be able to do at least as well as, or better than, they can if they are not able to adjust. Each firm can constantly monitor the other's activities and will tend to fine tune its expenditure according to its opponent's moves at each instant. It might be conjectured that, since the opponent knows it will provoke a response it will also tend to fine tune its expenditure to get the best payoff in the situation. If firms have to commit themselves at the beginning of the game and do not have the possibility of this fine tuning they each tend to overspend.

Details of the analysis are in Appendix 2. The solution gives the dynamics in terms of the function $\varphi_i(t)$ where φ_i has the same place in the dynamics as α_I in the previous equations for u and x. We have

$$\dot{\varphi} = \frac{3k}{2}\varphi^2 \tag{8}$$

and this can be used to get explicit solutions for the trajectories. It gives us more information on the properties of the competition, however, if we explore the difference between the way in which the firms behave in the open loop game where strategies are fixed and the closed loop game where they adjust their strategies at each instant.

<u>Comparison of Open and Closed Loop Strategies</u>

The comment that trajectories are essentially the same is made more rigorous by saying that the firms have the same profile if their expenditure moves in basically the same direction with the same acceleration at every instant. This requires that the first and second derivatives are the same. In the case where $x_1(0) = \frac{1}{2}$ this result is immediate from the equation for x in Appendix 2. If firm one has more initial support than firm two x is bounded away and above $x = \frac{1}{2}$ for all $t \in [0,1)$ in both games and hence must have the same qualitative properties. In addition $u_1 < u_2$ in both games. In order to get the rest of the profile compare equations (8) and (12).

It is also possible to get more details on the way in which the fraction of the potential pool of customers and expenditures change by looking at explicit solutions to the equations in Appendix 1 and Appendix 2. Some routine calculation shows that support for firm one is always a higher fraction of the available customers in the closed loop than in the open loop game but that the level is roughly equal at the end of the campaign. Since this is the payoff that matters the results of the campaign are more or less the same. Both firms spend less during the entire period.

THE GAME WITH FIRM ONE AS THE LEADER

The firms again formulate their strategies at the beginning of the game but it is assumed that firm one announces its plan first and then firm two formulates its strategy. It is assumed that firm one has more initial support than firm two. This might be thought of as a situation where firm one is the market leader and firm two a challenger that waits on firm one's actions. In this case we get similar trajectories to those in the first open loop game. If the impact cost for firm two is much lower than firm one support for firm one will be decreasing along the optimal path although there is a possible case where it decreases and then increases. It can also be shown that firm one is increasing its expenditure at the beginning and end of the time period.

Details of the analysis are in Appendix 3. This tells us that the firms should spend

$$u_1 = \alpha_3 \frac{k_1}{c_1}(1-x) + k_1\alpha_2\alpha_4 \text{ and } u_2 = -\alpha_2 \frac{k_2}{c_2}x \tag{9}$$

at each instant where the α terms have an analogous role to the solution in the first game.

In order to get the trajectory for the fraction of the customers going to firm one we substitute this into the equation for $\dot{x}$. In a similar manner to the previous analysis the dynamics of support is given by an equation with the relevant root written $\bar{r}$. For $x > \bar{r}$ we get $\dot{x} < 0$ and for $x < \bar{r}$ we get $\dot{x} > 0$ where $\bar{r}$ is some number such that $\dot{x} = 0$. Unlike r in the analysis of the open loop game, however, $\bar{r}$ increases as

t increases. This means that we cannot rule out the possibility that $\dot{x}$ switches sign and support for firm one starts to increase at some time.

Case 1: Equal Impact Costs

Support for firm one initially declines but may increase near the end of the time period. Firm two starts by spending more than firm one and increases its expenditure for all time and firm one also increases its expenditure for some time interval near the beginning of the period. For the specific case where $\dfrac{k_i}{c_i}=1$ firm one also increases its expenditure in an interval near the end.

Analysis of the values for $\bar{r}$ gives $\dot{x}(0)<0$. For some $x(0)$ sufficiently close to $1/2$ it must be the case that x is increasing in the vicinity of $x = 1$. This differs from the previous cases where the fraction of the resources for the larger firm was either monotonically increasing or decreasing over time. See figure 2 for an example.

Figure 2. Example of trajectory for $\dot{x}$ switching sign.
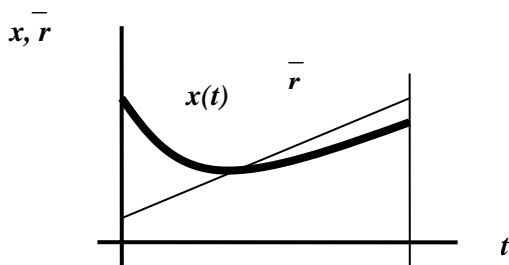
Some routine work using the costate values in Appendix 3 gives $u_2(0) > u_1(0)$ as in the open loop game and also $\dot{u}_2 > 0$ and $\ddot{u}_2 > 0$ for all t. We see that, as in the previous cases firm two is accelerating its expenditure for all time. If we consider the special case where $k_i = c_i = 1$ for $i = 1,2$ it is possible to show that firm one is accelerating its expenditure at the beginning and the end of the campaign. It seems plausible that it is accelerating its expenditure for all time, but I have not been able to develop a proof at this stage.

Case 2: Impact Cost of Firm One is Less Than Firm Two

In this case support for firm one is increasing for all time for κ sufficiently small. To see this observe that, in a similar manner to the analysis of r in the previous case, we can make $\bar{r}$ as close to one as we wish by letting $\kappa \rightarrow 0$. The trajectory of expenditure for firm two remains the same as in the previous case and accelerates for the complete period. If the coefficients are set at $k_i = c_i = 1$ the trajectory of expenditure for firm one is also the same at time $t = 0$ and $t = 1$ as in the previous case.

Case 3: Impact Cost of Firm One Greater Than Firm Two.

If $\kappa > m$ for some m sufficiently large we have $r \rightarrow \varepsilon$ for any $\varepsilon > 0$ and hence support for firm one is decreasing. Although firm one is losing support the trajectory of expenditures for firms one and two are the same as in Case 2.

CONCLUSION

This paper develops a simple model of the dynamics of a marketing competition in order to give some insights into the forces that shape the trajectory of expenditures. Its main finding is that, under the different information conditions studied, the firms mostly accelerate their expenditure for all time. This means that any observed acceleration in marketing effort can, in part, be explained by the dynamics of the struggle between the firms. Where there is asymmetry, the firm with the lower level of support tends to try harder. Its main limitation is, of course, that it involves a great deal of abstraction from reality.

It would be possible to extend this analysis in a number of ways that might make it more realistic. Among these are those in which the fraction of market shares could exhibit a jump discontinuity. It might be the case, for example, that customers are of different sizes or types, or that firms could choose a time in which the gave away free goods. Alternatively firms could be allowed to make a capital investment in a superior selling technology at a fixed price. It might also be possible to relax the assumption that the market size is fixed.

APPENDICES

The proofs are set out in a shortened form. Full details are available from the author.

Appendix 1: Open Loop Game

The problem is solved using the Pontryagin principle. The Hamiltonians are

$$H_1 = \frac{c_1}{2}u_1^2 + \alpha_1(k_1(1-x)u_1 - k_2 xu_2) \text{ and } H_2 = -\frac{c_2}{2}u_2^2 + \alpha_2(k_1(1-x)u_1 - k_2 xu_2) \tag{10}$$

where α_1 for $i = 1, 2$ are the costates and are required to be continuously differentiable. Since the Hessian matrices for the Hamiltonians are negative semi-definite the necessary conditions are also sufficient. The solution for the costates is

$$\alpha_1(t) = e^{-\int_t^1 k_1 u_1(t) + k_2 u_2(t)} \text{ and } \alpha_2(t) = -e^{-\int_t^1 k_1 u_1(t) + k_2 u_2(t)} \tag{11}$$

Solution for asymmetrical support $\kappa = 1$.

To analyze the trajectory of support for firm one, note that $sign \; \dot{x} = sign \; (1-2x)$ and hence $\dot{x} = 0$ for $x = \frac{1}{2}$ and x is bounded away from and above one half for all $t \in [0,1)$. This gives $\dot{x} < 0$ for all t. Substituting for $\frac{k_i^2}{c_i} = k$ into Equation (4) gives

$$\dot{\alpha} = k\alpha^2 \tag{12}$$

and this can be used to solve for α and give the required results.

Case 2. $\kappa < 1$.

To see why support for firm one increases for all t for κ sufficiently small note that for $\kappa = 0$ we have $r = 1$ and as κ increases r decreases. Taking limits $r \to \frac{1}{2}$ for $\kappa \to 1$ and hence $r \to 0$ for $\kappa \to \infty$ It follows that, for κ sufficiently small, $x(0) < r$ and $\dot{x} > 0$ for all t.

Appendix 2: Closed Loop Game

The Hamilton-Jacobi-Bellman equation is used to solve this problem. Write the value of the game for player i from time t and initial condition $x(0)$ as $\omega^i(t, x_o)$ and the partial derivative of ω^i with respect to any variable z as ω_z^i. This gives

$$-\omega_t^1 = \max_{u_1} (-\frac{c_1}{2} u_1^2 + \omega_x^1 (k_1 (1-x) u_1 - k_2 x u_2)) \tag{13}$$

with the analogous expression for ω_t^2. Solving this for u_1 and u_2 gives us the analogous expressions to (3) with the partial differentials ω_x^1 and ω_t^2 replacing α_1 and α_2 (Kamien and Schwartz, 1991, 259-63). Substituting the solutions back into Equation (13) and its counterpart and simplifying gives a system of two partial differential equations. Solving this gives

$$\dot{\varphi} = \frac{3k}{2} \varphi^2 \text{ which means } \varphi(t) = \frac{2}{2 + 3k(1-t)} \tag{13}$$

and substituting $\dot{x}$ into in Equation (7) and solving gives

$$x(t) = \frac{1}{2} (1 + (2x(0) - 1)(\frac{\varphi(0)}{\varphi(t)})^{\frac{4}{3}})$$

Appendix 3: Firm One as the Leader

The Lagrangean for firm one is

$$L_1 = \frac{c_1 u_1^2}{2} + \alpha_3 (k_1 (1-x) u_1 - k_2 u_2 x) + \alpha_4 (\frac{\partial H}{\partial x}) + \alpha_5 \frac{\partial H_2}{\partial u_2} \tag{14}$$

with H_2 given by Equation (10). $\alpha_4(t)$ is the costate associated with $\dot{\alpha}_2$ now treated as a state variable and $\alpha_5(t)$ is the multiplier for the condition that must hold for an optimum u_2. See (Basar and Olsder, 1995, 410-12). This gives us the necessary conditions for an internal solution as

$$u_1 = \alpha_3 \frac{k_1}{c_1} (1-x) + k_1 \alpha_2 \alpha_4 \text{ and } u_2 == \alpha_2 \frac{k_2}{c_2 x} \tag{15}$$

$$\dot{\alpha}_3 = (k_1 u_1 + k_2 u_2) \alpha_3 + k_2 \alpha_5 \alpha_2 \tag{16}$$

$$\dot{\alpha}_4 = -\alpha_4(k_1 u_1 + k_2 u_2) + k_2 \alpha_5 x \tag{17}$$

with transversality conditions $\alpha_3 = 1$ and $\alpha_4 = 0$ from (Basar and Olsder, 1995, 412). These conditions, and Equations (16) and (17) can be used to give the required results.

Analysis of $\dot{x}$

In order to establish the trajectory of support for firm one use equation (15) and the fact that $\alpha_1 \alpha_4 > 0$ to

get $\dot{x} = k_1(1-x)u_1 - k_2 xu_2 > \dfrac{k_1^2}{c_1}(1-x)^2 \alpha_3 - \dfrac{k_2^2}{c_2}x^2\alpha$ where $\alpha = -\alpha_2$ as before. Some work on this gives

$\bar{\dot{x}} > 0$ if

$$\overline{\varphi} = 1 - 2x + (1 - \kappa e^{\frac{k_2^2}{c_2}(1-t)})x^2 > 0$$

In a similar manner to the previous analysis of $\{x : \dot{x} = 0\}$ we get $\dot{x} < 0$ if $x > \bar{r}$ and $\dot{x} > 0$ where

$\bar{r} = (1 - \sqrt{\kappa e^{\frac{k_2^2}{c_2}(1-t)}})(1 - ke^{\frac{k_2^2}{c_2}(1-t)})^{-1}$. This gives similar results to the open loop game with the additional time dynamic given by the fact that, for k given, $\bar{r}$ increases as t increases. This means that we cannot rule out the possibility that $\dot{x}$ switches sign and support for firm one starts to increase at some time.

Analysis for $\kappa = 1$.

The positive root is now $\bar{r}(k,t) = \dfrac{1 - e^{\frac{\bar{k}(1-t)}{2}}}{1 - e^{\bar{k}(1-t)}}$. Taking the limit as $t \rightarrow 1$ gives $(\bar{k}, t) \rightarrow \dfrac{1}{2}$ and since

$\dfrac{\partial \bar{r}}{\partial t} > 0$ we have $\bar{r}(\bar{k}, 0) < \dfrac{1}{2}$. This means that $\dot{x} < 0$. For some $x(0)$ sufficiently close to ½ it must be the case that x is increasing in the vicinity of $x = 1$.

Analysis for u_1 and u_2 for $\kappa = 1$.
u_2 is analyzed by first noting from Equation (12) and the solutions to the costates that $\alpha(0) > \alpha_3(0)$. Since $\alpha_4(0) = 0$ we get $u_2(0) > u_1(0)$ and $u_2(0) > 0$ with $\ddot{u} > 0$ for all t.

To analyze the resource expenditure for firm one consider the special case where $k_i = c_i = 1$ for $i = 1,2$. Differentiating u_1 and simplifying gives $\dot{u}_1(0) > 0$ for all values of k_i and $c_1 : \kappa = 1$.

To get the sign for $\dot{u}_1$ at $t = 1$ use the solutions to the costates in Equations (16) and (17) to give $\dot{u}_1 > 0$ and $\ddot{u}_1(0) > 0$, $\ddot{u}_1(1) > 0$ as required.

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