A PUZZLE FOR TEACHING THE CONSTANT GROWTH STOCK PRICING MODEL

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ABSTRACT

The constant growth stock pricing model is an important component of introductory corporate finance courses, and an important step on the way to understanding the general two-stage model. In this paper, I present a relatively fun puzzle based on the relationships implied by the constant growth model. When solving this puzzle, students are forced to look beyond the apparent simplicity of the model, ensuring that they are ready to incorporate its concepts into more general situations. However, the most satisfying part of the exercise is the enthusiasm with which students approach the exercise, since, after all, it is a puzzle!

INTRODUCTION

The constant growth model is a straightforward approach to quantifying the "art" of stock pricing.[1] It is more intuitively appealing and apparently realistic than its simplest version, the constant dollar dividend model, but it still looks deceptively simple.[2] These features make it the model to which we appeal most often when a stock price is needed within the context of a larger problem (for example, when we value convertible bonds (Brigham and Gapenski, 1997)). Yet it is often misused, misunderstood, or underestimated, even by those of us who teach it. In this paper, I present a "puzzle" that has helped my students master the relationships implied by the constant growth model. By the time they are able to complete this puzzle, which many of them actually think is *fun*, they have demonstrated their appreciation for the requirements of the model (e.g., the growth rate must be "small") and their facility with its calculations. They are also prepared to quickly and easily grasp the general two-stage model.

The paper proceeds as follows. I first present a brief motivation for the model, which emphasizes its development in infinite time. I then outline the implications of the model. Finally, I give an example of the puzzle, with answers.

INTRODUCING THE CONSTANT GROWTH MODEL

This section provides an overview of a relatively painless way to lead students of introductory corporate finance through the development of the constant growth model.

The constant growth model is used to calculate a stock's fundamental price—that is, what the stock *should* sell for. What it *does* sell for is something entirely different. For us to bother with stock pricing at all, we must believe not only that we can identify a mispricing that can be exploited (for example, buying if the fundamental price exceeds the market price), but also that the market will eventually come to agree with us, eliminating the discrepancy. This is a tall order.

However, if we do choose to undertake this pricing exercise, we begin as we would with any other asset: noting that we never want to pay more for something than what we are going to get out of it. Thus:

$$\text{price} = \text{PV(all future cash flows)}. \qquad (1)$$

For stocks, this means that:

price = PV(dividends received during holding period and price received at sale), or

price = PV(dividends during holding period) + PV(price received at sale). (2)

All we need to do now is identify the amounts and timing of these cash flows.

Therein lies the rub. Unlike with bonds, we cannot simply look to the contract that exists to specify these cash flows. Instead, every input to a stock pricing model must be estimated. The presentation of stock pricing in an introductory finance course boils down to leading the students through this process of estimation.

We can start by making some plausible assumptions. (Of course, the quality of a model does not depend on the plausibility of its assumptions, which students should recognize.) First, assume that you are the only seller of a desirable stock. You have three types of interested buyers, who differ only by their desired holding periods: Zeb, who has an expected holding period of one year; Jessica, who plans to hold for five years; and Erik, who has no plans to ever sell (and to whom we therefore assign a holding period of infinity). Do you care about your potential buyers' holding periods? Does any seller ever care about the characteristics of his buyers? The answer to the latter question is "maybe": we can suppose, for example, that a conscientious breeder of registered puppies would not want to sell one to someone she deemed suspicious, or perhaps that the designer of a beautiful home may not want to sell it to someone with poor taste.[3] However, it is hard to imagine that the seller of a stock would feel emotionally attached to it, and so we suspect that you wouldn't care about your buyers' plans.

However, what if you did? It would not do you any good anyway, since you would be unable to price discriminate. Stocks are essentially commodities, and therefore your attempt to set different prices for different buyers would fall apart (through the development of a black market, for example).[4] Thus, our first assumption is that we cannot price discriminate when determining stock prices: all buyers pay the same price. This then means that price is independent of the buyer's holding period. This is extremely convenient, since it means we can develop our model in the easiest possible context—using the buyer with the most tractable holding period—then extrapolate that price to buyers with all holding periods. Given what we know about infinite series, that most tractable of holding periods will be Erik's: infinity.[5] This, then, is our final assumption: we can develop our model in infinite time.

Using an infinite holding period makes our model for price:

price at time t = PV(dividends from time [t+1] through time ∞), (3)

or

$$P_t = D_{[t+1]}/(1+i)^1 + D_{[t+2]}/(1+i)^2 + D_{[t+3]}/(1+i)^3 + \ldots, \quad (4)$$

where D_t is the dollar dividend paid at time t, and i is the market's periodic required rate of return for stocks of this risk class. This is the generic form of the stock pricing model, known as the dividend discount model (DDM).

Since the DDM is obviously intractable, we must posit various patterns for our infinite series of dividends. The constant growth model is the result of one such assumption: that dividends grow at the rate g% per period. That is:

$$D_{t+j} = D_t*(1+g)^j. \quad (5)$$

Substituting into the DDM, we have:

$$P_t = D_{[t+1]}/(1+i)^1 + D_{[t+1]}*(1+g)^1/(1+i)^2 + D_{[t+1]}*(1+g)^2/(1+i)^3 + \ldots, \quad (6)$$

which is a geometric series that converges to:

$$P_t = D_{[t+1]}/(i-g), \quad (7)$$

as long as $i > g$.[6] This is the constant growth model.

At this point, students should be reminded that the restriction that $i > g$ is not only necessary to make the series converge, but is also an economic requirement. The constant growth rate must be sustainable *to infinity*. The mathematical $i > g$ requirement should remind users that the constant growth model is for firms that are like parasites: parasites cannot continue to grow faster than their hosts, and companies cannot continue to grow faster than their economies. This is the point that is so often glossed over when this model is used cavalierly. It is also the reason that the derivation in infinite time is valuable, even though students may find all the infinity talk painful, especially when the final equation turns out to look so simple.

Now that students have the model, it is time to develop its implications and to apply this knowledge to the puzzle.

IMPLICATIONS OF THE CONSTANT GROWTH MODEL

Many of the "implications" mentioned here are really just rearrangements—different ways of looking at the constant growth expression. However, it is useful to point them out explicitly to students, who may otherwise overlook them.

(A) Price Grows at the Rate g Per Period, Too
Since $P_{[t+j]} = D_{[t+j+1]}/(i-g)$ and $D_{[t+j+1]} = D_{[t+1]}*(1+g)^j$, then $P_{[t+j]} = [D_{[t+1]}*(1+g)^j]/(i-g) = [D_{[t+1]}/(i-g)]*(1+g)^j = P_t*(1+g)^j$. Price gets pulled up by dividends, resulting in its growing at the same rate as dividends.

(B) Any Two Dividends or Prices Define the Constant Growth Rate, g
Since $D_{[t+j+1]} = D_{[t+1]}*(1+g)^j$, $g = [D_{[t+j+1]}/D_{[t+1]}]^{1/j} - 1$. The same goes for prices (and for earnings per share, for that matter).[7]

(C) The Required Rate of Return (i) Equals the Dividend Yield (dy) Plus g
The required return for any stock can be expressed as $(D_{t+1}/P_t) + [(P_{t+1} - P_t)/P_t]$, or [the dividend yield + the capital gains yield]. In all basic stock pricing models, we assume that i stays constant. However, we do not always assume that the components of i remain the same: in the general case, as long as their sum is i, they are free to change period by period. In the constant growth model, on the other hand, P_{t+1} is always equal to $P_t*(1+g)$, so that $(P_{t+1} - P_t)/P_t = [P_t*(1+g) - P_t]/P_t = g$. We can express this concept in many other ways:

(D) $i = dy + g$

(E) $dy = (i-g) = D_{[t+j+1]}/P_{[t+j]}$, for any t and j

(F) Capital Gains Yield = g

(G) The Dividend Yield For All Periods Is The Same

(H) The Capital Gains Yield For All Periods Is The Same

I will often summarize these points by giving students some tips for working through the puzzle:

- dividends get bigger over time, unless g=0
- prices get bigger over time, unless g=0
- given any two dividends, you can find g
- given any two prices, you can find g
- given a dividend and g, you can find any other dividend
- given a price and g, you can find any other price
- given a dividend and the prior period's price, you can find the dividend yield
- given dividend yield and g, you can find i
- i > g

AN EXAMPLE OF THE PUZZLE

There are four general types of statements that I use in the puzzle, each of which is illustrated in the example below. First, there are straightforward rate equalities (such as "i = 15%" or "dy = 12.25%"). Students usually start with these, trying to propose consistent sets of i, g, and dy values. Then there are algebraic manipulations of the rate relationships, which are often much more challenging. For example, if we have a statement of the form "$D_{12}/P_{17} = .1040987$," students would need to manipulate their various dy and growth rate values to solve for $dy/(1+g)^6$ (or actually solve for D_{12} and P_{17}). Third, there are cash flow statements giving amounts for various dividends or prices. These require students to test whether two dividends or prices can be linked with a specific growth rate, or whether a dividend and a price generate one of the known dividend yields. Finally, there are conceptual statements such as "this stock is overvalued if the market price is $50." These remind students about the whole point of stock pricing: we are drawing a distinction between the market price (what the price *is*) and the fundamental price (what the price *should be*), in order to identify trading opportunities.

By the time a student has worked through one or two examples of the puzzle, she should be fluent with all of the model's implications. I usually assign a puzzle after first having students do the most basic type of problem (such as: "Given that i=*x*, g=*y*, and P_0=*z*, find dy, P_{12}, D_6," etc.) and then going through an example puzzle in class. That has proved to be adequate preparation.

The Rules

Each of the statements below refers to one of three constant growth stocks, cleverly named A, B, and C. Determine which stock belongs to each statement.

The Puzzle

An example puzzle is presented in Figure 1.

Figure 1: The Puzzle

$dy = 7.95\% + g$	(1)		C	(13)	$P_3 = \$92.7495$
$D_7/P_6 = D_{12}/P_{11} = .1225$	(2)			(14)	$i = 15.00\%$
$D_1 = \$3.5775$	(3)			(15)	fundamental price = \$85.50
$[D_{10}/D_4]^{1/6} - 1 = .0275$	(4)			(16)	this stock is overvalued if market price = \$65.12
$D_{16}/i = \$45.00$	(5)			(17)	D_{14}/P_{13} for this stock = D_{14}/P_{13} for stock B
required return to equity = 9.20%	(6)			(18)	$P_{12}/(1.0125)^3 = P_9$
you should buy this stock if market price = \$150.75	(7)			(19)	$D_1 = \$10.4738$
$P_{11}*(.0795) = \$22.7852$	(8)		A	(20)	$P_4 = \$262.7363$
$P_0 = \$250.0000$	(9)			(21)	$(P_{13}/P_9) - 1 = .050945$
$(i-g) + [(P_{17} - P_{16})/P_{16}] = 7.95\%$	(10)			(22)	$D_6 = \$11.9953$
$D_{10} = \$13.3702$	(11)			(23)	$D_7 = \$3.5775$
$g + dy = 0.15$	(12)			(24)	$D_6/P_4 = 8.0494\%$

I always provide a detailed answer so that students can check their work or identify any problems that they are having. What follows is an example of an answer key for the puzzle above. (Please note that each statement above is numbered; those numbers will be used below to help us keep track of our work.)

The Answer

The easiest way to start this problem is to isolate all of the i, g, and dy values, then see if we can match them up. Toward that end, here are some notes about various statements:

dy statements

(2) $D_7/P_6 = D_{12}/P_{11} = .1225$

(8) $P_{11}*(.0795) = \$22.7852$

i statements

(6) required return to equity = 9.20%

(10) $(i-g) + [(P_{17} - P_{16})/P_{16}] = 7.95\%$

(12) $g + dy = 0.15$

(14) $i = 15.00\%$

g statements

(4) $[D_{10}/D_4]^{1/6} - 1 = .0275$

(18) $P_{12}/(1.0125)^3 = P_9$

From the i statements, we know that our three required returns are 9.2%, 7.95%, and 15%. The dividend yield statements tell us that two of our dy values are 12.25% and 7.95%. Finally, since g links two prices together, we can tell from the g statement that one of the stocks has a growth rate of 1.25%.

Here is another valuable statement:

(5) $D_{16}/i = \$45.00$

This is the stock-pricing model for no-growth stocks. It tells us that one of our stocks has g = 0% and P = \$45.00.

Given our rate information, we can match up the pieces as shown in Table 1.

Table 1: Required Return Component Matches

g	dy	i
0.00%	7.95%	7.95%
2.75%	12.25%	15.00%
1.25%	7.95%	9.20%

Now, we just need to know how to associate them with the specific stocks.

The easiest thing to do is to note that, since one of the stocks is in zero-growth, its price must stay constant at \$45.00. However, we are given other future stock prices for both A and C, so the \$45 stock must be B. This gives us:

(1) $dy = 7.95\% + g$	B
(5) $D_{16}/i = \$45.00$	B
(10) $(i\text{-}g) + [(P_{17} - P_{16})/P_{16}] = 7.95\%$	B

(Note that statement (1) works when g = 0%.) Now, if P = \$45 and dy = 7.95%, then D = \$3.5775, so:

(3) $D_1 = \$3.5775$	B
(23) $D_7 = \$3.5775$	B

Now that we have identified stock B as the no-growth stock, let us see if we can match the other values to A and C. We can start with C's P_3 of \$92.7495. If this is the stock that is growing at 2.75%, then we would expect a P_0 of $\$92.7495/(1.0275)^3 = \$85.50$. We have such a price! Now we know:

(2)	$D_7/P_6 = D_{12}/P_{11} = .1225$	C
(4)	$[D_{10}/D_4]^{1/6} - 1 = .0275$	C
(12)	g + dy = 0.15	C
(14)	i = 15.00%	C
(15)	fundamental price = $85.50	C

We also now know that stock A must be the one with the 9.20% required return:

(6)	required return to equity = 9.20%	A
(17)	D_{14}/P_{13} for this stock = D_{14}/P_{13} for stock B	A
(18)	$P_{12}/(1.0125)^3 = P_9$	A

A must also be the one with the last remaining P_0:

(9)	P_0 = $250.0000	A

Here is another A statement:

(8)	P_{11}*(.0795) = $22.7852	A

We can link this with A because it is saying "P_{11} times the dividend yield equals D_{12}, and D_{12} is $22.7852." B also has a dividend yield of 7.95%, but its price is always $45.00 and its dividend is always $3.5775.

Let us review our P_0 assignments and knock off some "under-" and "overvalued" statements. Table 2 summarizes our work so far.

Table 2: Summary of Assigned Stock Features

g	dy	i	P_0	Stock
0.00%	7.95%	7.95%	$ 45.00	B
2.75%	12.25%	15.00%	$ 85.50	C
1.25%	7.95%	9.20%	$250.00	A

Given these prices, we can see that:

(7)	you should buy this stock if market price = $150.75	A
(16)	this stock is overvalued if market price = $65.12	B

Now let us look at some manipulations of basic data. For example, statement (21) asks us to quantify the ratio (P_{13}/P_9). Since:

$$(P_{13}/P_9) - 1 \quad = \quad (1+g)^4 - 1, \qquad (8)$$

we can link this statement to one of the two positive growth rates we have by substituting each g into (8) and seeing if we get the 5.0945% result. We find a match using A's growth rate of 1.25%, so:

(21) $(P_{13}/P_9) - 1 = .050945$ — A

Now, here is a statement that involves a more difficult manipulation:

(24) $D_6/P_4 = 8.0494\%$

A dividend divided by a price is like a dividend yield, but the D is supposed to be *just after* the P. Thus, D_5/P_4 is a dividend yield, but D_6/P_4 is not. However, we can rearrange the given ratio as follows:

$$D_6/P_4 \quad = \quad D_5*(1+g)^1/P_4 \quad = \quad [D_5/P_4]*(1+g)^1 \quad = \quad dy*(1+g)^1. \qquad (9)$$

Thus, we are looking for a value just a bit bigger than our dividend yield. Again substituting in A's data, we find that (1.0795)*(1.0125) = 8.0494%, so:

(24) $D_6/P_4 = 8.0494\%$ — A

At this point, we are left with only these three cash flow statements:

(11) D_{10} = \$13.3702
(19) D_1 = \$10.4738
(22) D_6 = \$11.9953

Using what we know about dividend yield and P_0 to find D_1, we can assign these statements to stocks A and C, as shown in Table 3.

Table 3: Stocks' Initial Prices and Dividends

dy	P_0	D_1	Stock
7.95%	\$ 45.00	\$ 3.5775	B
12.25%	\$ 85.50	\$10.4738	C
7.95%	\$250.00	\$19.8750	A

Now we see that all of these dividends must belong to C! We can verify this by checking the growth rate implied by each pair of dividends:

$$(D_6/D_1)^{1/5} - 1 \quad = \quad 2.75\% \quad = \quad g \text{ for stock C} \qquad (10)$$

$$(D_{10}/D_1)^{1/9} - 1 \quad = \quad 2.75\% \quad = \quad g \text{ for stock C} \qquad (11)$$

$$(D_{10}/D_6)^{1/4} - 1 \quad = \quad 2.75\% \quad = \quad g \text{ for stock C.} \qquad (12)$$

Thus:

(11)	D_{10} = \$13.3702	C
(19)	D_1 = \$10.4738	C
(22)	D_6 = \$11.9953	C

That completes the puzzle!

<u>The Summary</u>

The completed puzzle is presented below in Figure 2.

Figure 2: The Completed Puzzle

dy = 7.95% + g	(1)	B	C	(13)	P_3 = \$92.7495
$D_7/P_6 = D_{12}/P_{11}$ = .1225	(2)	C	C	(14)	i = 15.00%
D_1 = \$3.5775	(3)	B	C	(15)	fundamental price = \$85.50
$[D_{10}/D_4]^{1/6} - 1 = .0275$	(4)	C	B	(16)	this stock is overvalued if market price = \$65.12
D_{16}/i = \$45.00	(5)	B	A	(17)	D_{14}/P_{13} for this stock = D_{14}/P_{13} for stock B
required return to equity = 9.20%	(6)	A	A	(18)	$P_{12}/(1.0125)^3 = P_9$
you should buy this stock if market price = \$150.75	(7)	A	C	(19)	D_1 = \$10.4738
P_{11}*(.0795) = \$22.7852	(8)	A	A	(20)	P_4 = \$262.7363
P_0 = \$250.0000	(9)	A	A	(21)	$(P_{13}/P_9) - 1 = .050945$
$(i-g) + [(P_{17} - P_{16})/P_{16}]$ = 7.95%	(10)	B	C	(22)	D_6 = \$11.9953
D_{10} = \$13.3702	(11)	C	B	(23)	D_7 = \$3.5775
g + dy = 0.15	(12)	C	A	(24)	D_6/P_4 = 8.0494%

CONCLUSIONS

Understanding the constant growth case is a prerequisite to understanding the general two-stage model. However, many textbook end-of-chapter problems on basic stock pricing do not rigorously test the students' knowledge of the relationships inherent in the constant growth case, so that students may be inadequately prepared for more difficult problems. For example, of 25 questions at the end of the "Equity Markets and Stock Valuation" chapter in *Essentials of Corporate Finance* by Ross, Westerfield, and Jordan (2007), thirteen involve the constant growth model. The structure of these questions can be mapped as shown in Table 4.

Table 4: Examples of Constant Growth Stock Pricing End-of-Chapter Questions

Inputs Given					Answers to Find
	D_0	g		i	P_0
	D_0	g		i	P_0
	D_1	g		i	P_0
	D_1	g		i	P_0
P_0	D_0	g			i
P_0	D_0	g			i
P_0	D_1	g			i
P_0	D_1	g			i
P_0	D_1	g		i	dy & cgy
P_0	D_1			i	g
P_0		g	dy	i	D_0
P_0		g		i	D_0
		g	dy		i

No question asks a student to consider more than one of the model's implied relationships for any given stock. However, forcing students to go beyond simple plug-and-chug exercises helps them appreciate the implications of the deceptively simple $D_1/(i-g)$ ratio. This appreciation will make it less likely that they will misuse the model (by assuming that it is appropriate for a company currently growing at 11% per year, for example) and more likely that they will more easily master the two-stage model.

In this paper, I present a puzzle that does force students to work more deeply with a few stocks, and to explicitly acknowledge the constant growth model's implications for those stocks. This puzzle has prepared my students well for the relative rigors of the two-stage model. More importantly for them, students often actually enjoy working through these exercises. They tell me that they are grateful for any spoonful of sugar that makes this fairly difficult course go down more easily.

ENDNOTES

1. This model is also called the "Gordon growth model" or the "Gordon/Shapiro model." See, for example, Brigham and Gapenski (1997), p. 309, and Brealey, Myers, and Allen (2006), p. 65.

2. The constant-dollar model is a special case of the constant growth model—the case of g=0%. However, the constant-dollar model is usually introduced as a separate model, before the more general constant growth model is presented.

3. These are examples I've encountered in real life.

4. Mansfield [1994] identifies the necessary conditions for third-degree price discrimination to be (1) "considerable differences in the price elasticity of demand" for the product, perhaps caused by differences in tastes or lack of substitutes; (2) the seller's low-cost ability to identify and segregate these groups; and (3) the inability for buyers to transfer the good among themselves. While we could perhaps assume that different expected holding periods could create differences in "tastes," we would still have to acknowledge that any particular issue of stock has many substitutes. As for the second and third requirements, we hold out even less hope for successful discrimination.

5. It is fun to ask students to guess whose holding period will be the most tractable. Zeb is usually their answer.

6. An infinite series of the form "sum = A + A*B + A*B^2 + A*B^3 + …" is a geometric series. These series converge to A/(1-B) if |B| < 1. In the constant growth case, A = D_1/(1+i), and B = (1+g)/(1+i), which is less than 1 if i > g.

7. The payout ratio is constant under the model's assumptions.

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