THE APPLICABILITY OF BENFORD'S LAW TO THE BUYING BEHAVIOR OF FOREIGN MILITARY SALES CUSTOMERS

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ABSTRACT

The forces of natural law selection prompt animal species to make preference decisions to maximize their survival utility. Those species making decisions in this way survive, and those which do not tend towards extinction. This process appears to follow a decreasing marginal utility trend with a logarithmic probability distribution, which is identical to Benford's Law. This suggests that Benford's Law is a descriptive statistic of this natural selection process. Additionally, the customers of the Defense Security Cooperation Agency's foreign military sales program presumably also attempt to optimize their survival utility, and exhibit purchasing patterns correlating strongly with Benford's Law. The purpose of this paper is to examine how a mathematical phenomenon, Benford's Law, may prove to be a useful means of understanding the buying behavior of DSCA's foreign customers. This paper also suggests an informal proof of Benford's Law.

JEL: C44, M31

INTRODUCTION

The need to consume is largely driven by the desire to satisfy needs that have been programmed in our minds either by the genes we inherit or the memes we learn from the culture in which we live. To understand why people make preference decisions, examining a taxonomy developed by Abraham Maslow would be a good starting point. Csikszentmihalyi (2007) suggested that according to Maslow's theory, the most basic needs that motivate a person are physiological survival needs: to eat, to drink, to have sex, to breathe, to sleep, to be warm, and to eliminate. When these needs are not met, the person will turn all of his or her psychic energy to the task of satisfying them.

The author said that if the physiological needs are relatively well gratified, there then emerges a new set of needs, which are categorized as safety needs, love and belonging needs, and esteem and self-actualization needs. This paper focuses on making preference decisions to maximize an individual's survival utility. Survival utility is seen as an active and dominant mobilizer of an individual's resources in times of war, disease, natural catastrophes, crime waves, and societal disorganization.

Therefore, the purpose of this paper is to examine how a mathematical phenomenon, Benford's Law, might prove to be a useful means of understanding the buying behavior of consumers. We apply this in particular to the Defense Security Cooperation Agency (DSCA) -- which is an organization within the US Department of Defense that facilitates sales of certain weapons, military equipment, and military training to customer countries allied with the US. We will first review the literature to introduce Benford's Law, and relate why Benford's Law initially appears to be enigmatic. Then, we will discuss research regarding several concepts of stimulus and response, and how they are mathematically modeled. We believe that these research findings show the cause of the underlying Benford's Law distribution. Since rational customers buy according to stimulus and response effects, we test the prediction that DSCA customers will buy according to the Benford's Law distribution. Finally, we show the statistical significance of our research.

LITERATURE REVIEW

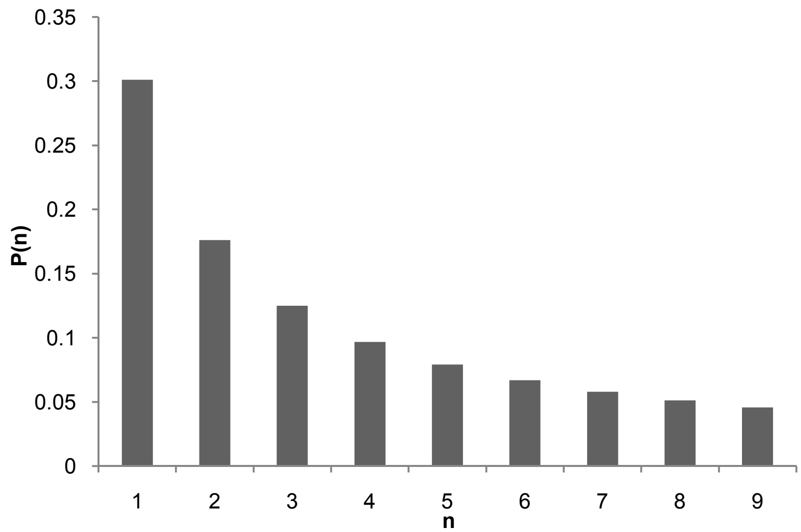
Benford's Law, also called the first-digit law, states that in lists of numbers from many real-life sources of data, the leading digit 1 occurs more often than the others, namely about 30% of the time. A lively account is from a blog entitled Fabulous Adventures in Coding.

> *While I was poking through my old numeric analysis textbooks to refresh my memory for this series on floating point arithmetic, I came across one of my favorite weird facts about math. A nonzero base-ten integer starts with some digit other than zero. You might naively expect that given a bunch of "random" numbers, you'd see every digit from 1 to 9 about equally often. You'd see as many 2's as 9's. You'd see each digit as the leading digit about 11% of the time. For example, consider a random integer between 100000 and 999999. One ninth begin with 1, one ninth begin with 2, etc. But in real-life datasets, that's not the case at all. If you just start grabbing thousands or millions of "random" numbers from newspapers and magazines and books, you soon see that about 30% of the numbers begin with 1, and it falls off rapidly from there. About 18% begin with 2, all the way down to less than 5% for 9. This oddity was discovered by Newcomb in 1881, and then rediscovered by Frank Benford, a physicist, in 1937. As often is the case, the fact became associated with the second discoverer and is now known as Benford's Law. Benford's Law has lots of practical applications. For instance, people who just make up numbers wholesale on their tax returns tend to pick "average seeming" numbers, and to humans, "average seeming" means "starts with a five." People think, I want something between $1000 and $10000, let's say, $5624. The IRS routinely scans tax returns to find unusually high percentages of leading 5's and examines those more carefully. Benford's result was carefully studied by many statisticians and other mathematicians, and we now have a multi-base form of the law. Given a bunch of numbers in base B, we'd expect to see leading digit n approximately in (1 + 1/n) / in B of the time. But what could possibly explain Benford's Law? (Lippert 2005)*

Browne (1998) said that Benford's Law is named for the late Dr. Frank Benford, a physicist at the General Electric Company. The author indicated that in 1938 Dr. Benford noticed that pages of logarithms corresponding to numbers starting with the numeral 1 were much dirtier and more worn than other pages. Browne (1998) defined a logarithm as an exponent. He said that any number can be expressed as the fractional exponent, the logarithm, of some base number, such as 10. Published tables permit users to look up logarithms corresponding to numbers, or numbers corresponding to logarithms. Logarithms tables, and the slide rules derived from them, are not much used for routine calculating anymore, electronic calculators and computers are simpler and faster. But logarithms remain important in many scientific and technical applications, and they were a key element in Dr. Benford's discovery.

More formal treatment of Benford's Law appears in the literature. A typical treatment is from Kollath-Romano (2004). The presentation includes a histogram showing the distribution of the leading digits copied below in Figure 1. The literature has identified many possible practical applications for Benford's Law. Nigrini (1999) reported the following listing of applications: accounts payable data; estimations in the general ledger; the relative size of inventory unit prices among locations; duplicate payments; computer system conversion (for example, old to new system; accounts receivable files); processing inefficiencies due to high quantity/low dollar transactions, new combination of selling prices; and customer refunds. The author went on to say that Benford's Law provides a unique method of data analysis that could help alert Certified Public Accountants to spot irregularities indicating possible error, fraud, manipulative bias or processing inefficiency. Nigrini appears to be the first researcher to apply Benford's Law extensively to accounting numbers with the goal to detect fraud (Durtschi, Hillison and Pacini, 2004).

Figure 1: Graph from Kollath-Romano



This figure shows the distribution of leading digits n as forecast by Benford's Law.

Durtschi, Hillison and Pacini (2004) said that Benford's Law has been promoted as providing the auditor with a tool that is simple and effective for the detection of fraud. Moreover, Benford's Law as applied to auditing is simply a more complex form of digital analysis. It looks at an entire account to determine if the numbers fall into the expected distribution. The authors concluded that Benford's Law is a useful tool for identifying suspect accounts for further analysis Skousen, Guan and Wetzel (2004) reported that in using Benford's Law their research provided evidence that managers of Japanese firms tend to engage in earnings manipulative activities of rounding earnings numbers to achieve key reference points. The authors suggested that the first digit of earnings numbers was often emphasized by the management.

Two researchers reported that Benford's Law is seeing increasing use as a diagnostic tool for isolating pockets of large datasets with irregularities that deserve closer inspection (Tam Cho and Gaines, 2007). Varian (1972), an economist, reported that Benford's Law can be used as a test of the honesty or validity of purportedly random scientific data in a social science context. Although the literature reveals numerous studies describing Benford's Law and characterizations of its mathematical patterns, no study was found either examining the applicability of Benford's Law to customer buying behavior or giving causes for its occurrence.

THEORY AND METHODOLOGY

This section of the paper discusses two underlying research studies that provide some of the rationale behind Benford's Law. These studies include Weber's Study of the Perception of Human Stimuli and The Decreasing Utility Function of Natural Law.

Weber's Study of the Perception of Human Stimuli

This study of the underlying causes of Benford's Law begins with the work of Ernst Heinrich Weber. Weber (2005) found a form of the law of diminishing returns relationship in humans between stimulus and response: as stimulus increased, response also increased but at a decreasing rate that can be modeled logarithmically. The resulting Weber-Fechner Law indicates that this response utility increases logarithmically.

In one of his classic experiments, Weber gradually increased the weight that a blindfolded man was holding and asked him to respond when he first felt the increase. Weber found that the response was proportional to a relative increase in the weight. That is to say, if the weight is 1 kg, an increase of a few grams will not be noticed. Rather, when the mass is increased by a certain factor, an increase in weight is perceived. If the mass is doubled, the threshold is also doubles. This kind of relationship can be described by a differential equation as:

$$dp = k\frac{dS}{S} \tag{1}$$

Where dp is the differential change in perception, dS is the differential increase in the stimulus and S is the stimulus at the instant. It is instructive to look at this equation. Dividing both sides by dS yields the following:

$$\frac{dp}{dS} = k * \frac{1}{S} \tag{2}$$

This is the form for the equation for the natural logarithm ("log"), where the slope of the line equals 1/x. In this case, the "x" axis is the change in stimulus S, and the "y" axis is the change in response (perception p). The equation could also be written in the following form:

Response Level = k * ln Stimulus Level + C

where k is a constant and C the constant of integration. Natural logs can be converted into base 10 logs which may be more familiar. Table 1 shows this conversion. Table 1 shows that when the Stimulus Level doubles (i.e., it increases from 1 to 2 – see column 1), the Response Level increases by a factor of the log of 2 (i.e., .3010, or 30.10% --see column 3). Changing the Stimulus Level from 2 to 3 yields in incremental response increase of .1761 (i.e., .4771 - .3010), or 17.61%. One way to picture this is to recall the slide rule. Slide rules have several scales, for example, K, A, B, S, T, CI, C, D, and L. The L scale is the log scale. It starts with 0, and increases to the right in major increments of .1 until it reaches the right edge at 1. The integers on the D scale above it can be thought.

Table 1: Weber-Fechner Law of Stimulus and Response

Stimulus Level	Response Level Natural Log	Response Level Log (Base 10)	Incremental Increase	% Incremental Increase
1	0.0000	0.0000	0.3010	30.10%
2	0.6931	0.3010	0.1761	17.61%
3	1.0986	0.4771	0.1249	12.49%
4	1.3863	0.6021	0.0969	9.69%
5	1.6094	0.6990	0.0792	7.92%
6	1.7918	0.7782	0.0669	6.69%
7	1.9459	0.8451	0.0580	5.80%
8	2.0794	0.9031	0.0512	5.12%
9	2.1972	0.9542	0.0458	4.58%
10	2.303	1.000	--	--

This table shows the logarithmic, not linear, relationship between stimulus and response.

of as corresponding to the Stimulus Levels of Table 1. Their logs are directly below them on the L scale, and these can be thought of as the Response Levels of Table 1, especially column 3. For example, the log of 1 (on the D scale) is 0 (on the L scale). This means that if the Stimulus Level stays constant, then the

Response Level does not change. The log of 2 (on the D scale) is close to .3010 (on the L scale). So, increasing the Stimulus Level by a factor of 2 increases the Response Level by a factor of .3010, or 30.10%. This is also the incremental increase in the fourth column of Table 1 from 1 to 2. Moreover, the distance from 1 to 2 on the D scale is 30.10% of the D scale's length.

Table 1 shows that increasing the Stimulus Level by a factor of 3 increases the Response Level by a factor of .4771 (or, 47.71%); .4771 on the L scale is directly below 3 on the D scale. Also, increasing the Stimulus Level from 2 to 3 corresponds to the difference between log 3 and log 2, or, .4771 - .3010, which is .1761 or 17.61%. Put another way, the distance from 2 to 3 on the D scale is 17.61% of the D scale's length. Further continuing, each additional integral increment on the D scale, such as from 3 to 4, will correspond to a percentage of the length of the D scale which can be matched to the corresponding incremental distance on the L scale. These all are the remaining numbers in column 4 of Table 1, respectively.

Those familiar with slide rule use know that although the D scale's integers run from 1 through 10, they can also be thought of as running from 10 to 100, 100 to 1,000, 1,000 to 10,000, or across any factor of 10. For example, the log of 2 is .3010, the log of 20 is 1.3010, the log of 200 is 2.3010, etc. The ".3010" factor (the mantissa) stays the same. If one wants a more complete understanding of Weber-Fechner Law of stimulus and response, then, one needs to recognize that increasing the stimulus from 10 to 20, or from 100 to 200, or from 1,000 to 2,000, etc. always increases the response by the mantissa .3010.

Suppose one runs a thought experiment with a statistically large number of trials of randomly distributed stimuli ranging over many powers of 10, and suppose that the corresponding Response Level results match those predicted by Weber-Fechner Law. Then, one counts the number of times the Response Level increased by the factor of .3010. That would be the number of times the Stimulus Level increased from 1 to 2, plus the number of times it increased from 10 to 20, plus the number of times it increased from 100 to 200, etc. Since the Response Level increases by .3010 in all cases, it does not matter if a particular trial was a 1 to 2, or a 10 to 20, or a 100 to 200 -- in this sense: in each trial, the leading digit of the beginning degree of stimulus would be a 1 (1, 10, 100, etc.). Therefore, during a large number of trials, the leading digit of 1 should have appeared about 30.10% of the time. Since increasing the Stimulus Level from 2 to 3 incurs a Response Level increase of 17.61%, the number of times the number 2 appears as the leading digit of randomized beginning degree of stimuli should be about 17.61% of the time. In like manner, 3 should appear 12.29% of the time, 4 should appear 9.69% of the time, and so on until 9 should appear 4.58% of the time. These are the same values as in Table 1, Column 4.

<u>The Decreasing Utility Function of Natural Law</u>

Hans-Werner Sinn (2002) extends the Weber-Fechner Law to all surviving species, as his overviews in the abstract of his paper, "Weber's Law and the Biological Evolution of Risk Preference." The paper offers proof that the expected utility maximization with logarithmic utility is a dominant preference in the biological selection process in the sense that a population following any other preference for decision-making under risk, with a probability that approaches certainty, disappear relative to the population following this preference as time goes to infinity. This result is contrasted with Weber's and Fechner's Psychophysical Law which implies sensation functions for objective stimuli (Sinn 2002).

Basically, Sinn argues that the forces of *natural selection* prompt animal species to make preference decisions to maximize their survival utility. Those species making decisions in this way survive, and those which do not tend towards extinction. This process follows a decreasing marginal utility trend with a logarithmic probability distribution. This is the same as Weber-Fechner's utility function described by Table 1. In other words, doubling the work effort results in an increase of survivability of .3010, or log 2, or, doubling the Stimulus Level results in a Response Level increase of .3010. These are the same numbers as in Table 1, especially column 4.

Benford's Law

Benford's Law, also called the first-digit law, states that in lists of numbers from many real-life sources of data, the leading digit 1 occurs more often than the others, namely about 30% of the time. The exact frequency of the leading digits is in Table 2.

Table 2: Benford's Law Prediction of Frequency of Occurrence of Leading Digit

Leading (First) Digit	Probability of Occurrence
1	0.3010
2	0.1761
3	0.1249
4	0.0969
5	0.0792
6	0.0669
7	0.0580
8	0.0512
9	0.0458

This table shows that the frequency of leading digits predicted by Benford's Law is identical to the stimulus and response level leading digits of the Weber-Fechner Law in Table 1.

Table 2 is of interest because the "Probability of Occurrence" of leading digits, as predicted by Benford's Law, is identical to the probability of leading digits as derived from the Weber-Fechner Law in column 4 of Table 1. This prompts the notion that Benford's Law appears to be a descriptive statistic of the characteristics of human behavior and natural selection as described, respectively, by the research of Weber and the research of Sinn. Benford's Law appears to be based, at least, upon the forces of natural selection which prompt animal and human species to make preference decisions to maximize their survival utility. Sinn argues that those species making decisions in this way survive, and those which do not tend towards extinction.

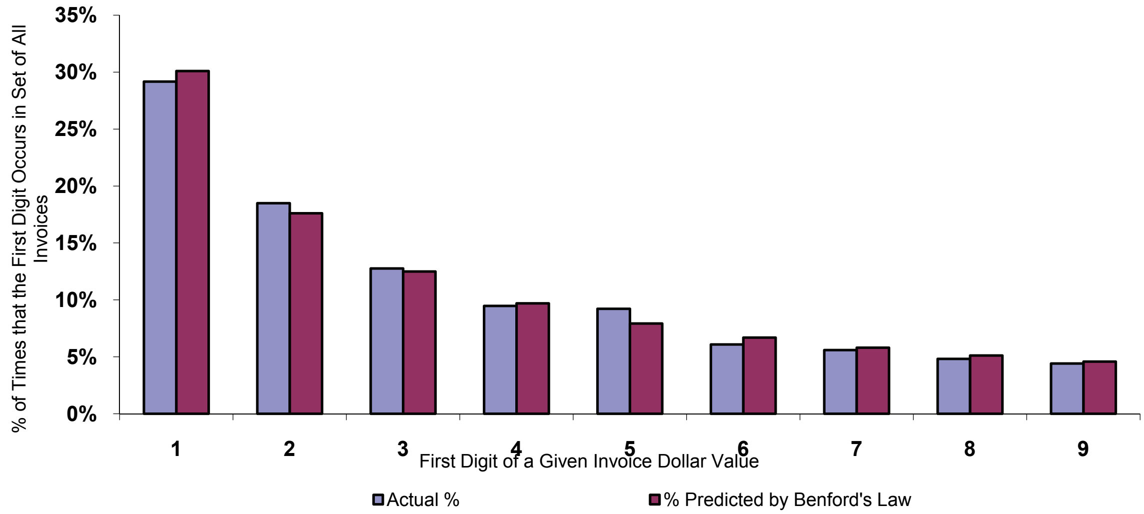
It is assumed by the researchers that when DSCA customers purchase goods, they receive a degree of utility in return for their money. This can be thought of as their perception of increased security utility. Increase the dollar value of the customer's order, and the customer receives increased security utility. However, customer security utility improvement does not increase in a linear fashion. Because military expenditures occur to improve a country's perception of increased security, it follows the forces of natural selection described by Sinn. For example, suppose that a customer country doubles its expenditure for military equipment. Their perception of increased security should increase by .3010, as predicted by both Weber-Fechner's Law and Benford's Law.

Testing this assumption requires at least two steps. First, we determine if there is a strong enough correlation between the actual distribution of leading digits on DSCA sales invoices and the distribution as predicted by Weber-Fechner Law and Benford's Law. Secondly, if there is such a strong enough correlation, then contact customer countries to determine if their actual perceptions closely match those determined statistically through the first step's correlation. This research is limited to Step 1.

DATA

The researchers downloaded a census of all of the DSCA invoices over a five year period, and the number of invoices was statistically large. Although the values of individual invoices are proprietary, Figure 2 shows the distribution of those invoices' first digits.

Figure 2: Actual Distribution of a Large Number of Invoice First Digits



Visual inspection of Figure 2 makes it very tempting to argue that the invoice first digit distribution approaches the above logarithmic distribution in Table 1, column 4, as a mathematical limit. The distribution of DSCA's invoice first digit percentages with those percentages predicted by the Weber-Fechner Law and Benford's Law is almost identical. The distribution of actual percentages with those percentages predicted by the Weber-Fechner Law and Benford's Law is almost identical. To measure the statistical significance of this comparison, one can use the chi-square test. The results of this test of comparison of actuals to predicted are shown in Table 3.

Table 3: Chi-Square Test for Actual Frequency of Sales Invoice First Digits

First Digit	Actual %	Predicted %	Difference	Squared Difference	Squared Difference / Predicted
1	0.292	0.301	-0.009	$8.15*10^{-5}$	$2.71*10^{-4}$
2	0.185	0.176	0.009	$7.94*10^{-5}$	$4.51*10^{-4}$
3	0.128	0.125	0.003	$9.37*10^{-6}$	$7.50*10^{-5}$
4	0.095	0.097	-0.002	$3.65*10^{-6}$	$3.76*10^{-5}$
5	0.092	0.079	0.013	$1.64*10^{-4}$	$2.08*10^{-3}$
6	0.061	0.067	-0.006	$3.54*10^{-5}$	$5.28*10^{-4}$
7	0.056	0.058	-0.002	$3.97*10^{-6}$	$6.84*10^{-5}$
8	0.048	0.051	-0.003	$9.94*10^{-6}$	$1.94*10^{-4}$
9	0.044	0.046	-0.002	$3.09*10^{-6}$	$6.75*10^{-5}$
Total					$\mathbf{3.77*10^{-3}}$

This table shows that the agreement between the actual leading digit frequencies of DSCA invoices and that forecast by Benford's Law is statistically significant.

The bottom line of this test is that the total deviation between actuals and predicted as measured by chi-square is very small – $3.77*10^{-3}$. This total is better (smaller) than the critical value of 1.344 for 99.5% confidence, which is the highest level of confidence in a university statistics book (Weiss 1999), so we at least initially perceive that we are more than 99.5% confident that Benford's Law correctly describes the frequency of our first invoice digits in this census. Therefore, it seems that DSCA sales patterns strongly enough correlate with Benford's Law.

RESULTS AND CONCLUDING COMMENTS

According to the Weber-Fechner study, humans respond to stimuli in a logarithmic manner. For example, when the stimulus is doubled, the response increases by about 30.10%, which corresponds to the logarithm of 2; if the stimulus increases from 2 to 3, then the additional response increases by about 17.71%; their sum of about 47.71% corresponds to the logarithm of 3. Sinn believes that this is in accordance with natural law, and that species that follow this law for survival remain, and those that do not follow this law become extinct.

What follows is that if stimuli levels are uniformly randomly distributed between 0 and 1, then the corresponding responses are logarithmically distributed. This means that about 30.10 % of the responses will have a leading digit between 1 and 2, about 17.71% of the responses will have a leading digit between 2 and 3, etc. This can be pictured using a slide rule. The same applies to stimuli uniformly randomly distributed between 1 and 10, 10 and 100, etc. because the mantissa
of those logarithms are identical to those between 0 and 1. We believe that there is a descriptive statistic that portrays this logarithmic distribution, and that is Benford's Law.

We found that DSCA customers seem also to respond in this way. The set of their invoice values are randomly distributed in this way, and this pattern follows the descriptive statistic of Benford's Law. We believe that at least one main reason for this is their desire to improve their perceptions national security as a response to their purchases. We believe that DSCA can use this information to help facilitate security cooperation in the international allied community.

We expect Benford's Law to be a statistic that describes a variety of situations similar to that experienced in DSCA. These are consumer buying behaviors where the customers' perception of satisfaction is a main effect of the purchase stimulus, there are a large number of customer orders, there is a wide variety of pricing levels, and customer order sizes cover several orders of magnitude in dollar size. Examples of these might be the size distribution of a statistically large number of software applications in a company's software portfolio as measured in function points; the distribution of perhaps a month of sales orders from a department store; or the quarterly sales orders at a gasoline service station to include parts, repair work, food, and gasoline pump sales. We also suggest that other organizations examine their sales invoice distributions for Benford's Law, and if they find it then they can examine whether their customers are also motivated to buy based on the Weber-Fechner and Sinn studies.

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