APPLICATION OF QUEUING THEORY TO DYNAMIC VEHICLE ROUTING PROBLEM

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ABSTRACT

In this paper, we developed and analyzed a dynamic model of the vehicle routing problem. In the stated model, a vehicle with adequate volume travels at a constant velocity in a bounded plane to provide services to independent and uniformly distributed demands. The dynamic demands arrive according to a Poisson process and on-site service times are generally distributed, independent of their location. A median repositioning policy for the dynamic model is proposed to reduce system time of the First Come First Served policy. The improvement of performance is verified by simulation results.

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KEYWORDS: Queuing Theory; Dynamic Vehicle Routing Problem

INTRODUCTION

The routing and scheduling of vehicles represents an important component of many distribution and transportation systems costs. The vehicle routing problem (VRP) involves the design of a set of minimum cost routes, originating and terminating at a central depot, for a vehicle that services a set of customers with known demands. Each customer is serviced exactly once and, furthermore, all the customers must be assigned to vehicles such that the vehicle capacity is not exceeded. The generic VRP and many of its practical occurrences have been studied intensively in the literature (Bodin et al. (1983), Magnanti (1981), and Laporte and Nobert (1987)). The vast majority of the vehicle routing literature is dedicated to deterministic and static models where all problem parameters are assumed to be known in advance and decisions are made only once at the beginning of the planning horizon.

However, distribution operations in practice often involve uncertainties with respect to locations and sizes of customer orders, vehicle travel times, etc. While in the past, research mainly concentrated on static variants of VRP where increasing interest in dynamic VRPs can be observed in recent years. Dynamic vehicle routing with previously unknown customer requests arriving over time has become increasingly important in the transportation industry as new technologies such as global positioning systems and wireless communications allow the assignment of new requests to vehicles in real time. In this research, the objective of the system is to minimize the expected customer waiting time and develop a dynamic VRP real time optimal policy. By computing the expected system time and through simulated examples, we intend to show that this policy outperformed the conventional first come first served (FCFS) policy.

The remainder of this paper is organized as follows. First, section 2 provides a brief literature review and general model dedicated to dynamic vehicle routing problems. In section 3 the real time service problem is introduced with existing and proposed models. Section 4 contains a mathematical simulation of the model. Conclusions and suggestions for future work are outlined in section 5.

LITERATURE REVIEW AND BACKGROUND

Vehicle routing problems are widely present in today's industries, ranging from distribution problems to fleet management. They account for a significant portion of the operational costs of many companies. Operations research techniques have been used with success in many situations for reducing such costs.

Even if most real instances of vehicle routing problems are solved with heuristic methods, the desire to produce optimal solutions has given rise to a prolific research area. Most results relating to dynamic vehicle routing models were published in the last 15 years. Surveys of existing results can be found in Powell (1988), Powell, Jaollet, and Odoni (1995), Psaraftis (1995), Bertsimas and Simchi-Levi (1996), and Gendreau and Potvin (1998). Compared with static VRP models, characteristics of dynamic VRP models include: (1) information is partially known when planning vehicle routes; (2) new information will arrive during the planning and execution process, and original information may alter; (3) it is not possible to obtain an ascertained route through a single attempt. Below are the general backgrounds and model definition for the dynamic VRP.

Geometrical Probability

Given two independently and uniformly distributed random points $z_1(x_{z_1}, y_{z_1})$, $z_2(x_{z_2}, y_{z_2})$ in a square of area A and length of a. To find the mean and variance of $|z_1, z_2|$, we let z_1 be the fix point $z^*(x_1, y_1)$, then

$$E\left(\left|z^* z_2\right|^2\right) = \iint_A \left(\left(x_{z_2} - x_1\right)^2 + \left(y_{z_2} - y_1\right)^2\right) dx_{z_2} dy_{z_2}$$
$$= \int_{-a/2}^{a/2} dx_{z_2} \int_{-a/2}^{a/2} \tfrac{1}{a^2}\left(\left(x_{z_2} - x_1\right)^2 + \left(y_{z_2} - y_1\right)^2\right) dy_{z_2}$$
$$= \left(\frac{a^2}{6} + x_1^2 + y_1^2\right)$$

Now, replacing z^* by z_1,

$$E\left(\left|z_1 z_2\right|^2\right) = \int_{-a/2}^{a/2} dx_{z_1} \int_{-a/2}^{a/2} \tfrac{1}{a^2}\left(\frac{a^2}{6} + x_{z_1}^2 + y_{z_1}^2\right) dy_{z_1},$$

we have

$$E\left(\left|z_1 z_2\right|^2\right) = \frac{a^2}{3} \tag{1}$$

Similarly, we can obtain

$$E\left(\left|z_1 z_2\right|\right) \approx 0.52\, a \tag{2}$$

$$Var\left(\left|z_1 z_2\right|\right) = E\left(\left|z_1 z_2\right|^2\right) - E\left(\left|z_1 z_2\right|\right)^2 \approx 0.06\, a^2. \tag{3}$$

Let z be a uniformly distributed random point in area A, z_0 be the median of A. By the above method, we have

$$E\left(\left|z_0 z\right|^2\right) = \frac{a^2}{6} \tag{4}$$

$$E\left(\left|z_0 z\right|\right) \approx 0.38\, a \tag{5}$$

$$Var\left(\left|z_0 z\right|\right) = E\left(\left|z_0 z\right|^2\right) - E\left(\left|z_0 z\right|\right)^2 \approx 0.02\, a^2 \tag{6}$$

Pollaczek-Khinchin (P-K) Formula for M/G/1 Queuing Model

For a queuing model, defined

λ = Probability of a customer arrived in one unit time, known as probability intensity

$\bar{s}$ = Average customer service time

$\rho =$ Service intensity, $\rho = \lambda \bar{s}$
$w =$ Average waiting time
$T =$ Average system time, $T = w + \bar{s}$
$N =$ Average number of customer in the system, including customers being served and those waiting for service, according to Little's formula, $N = \lambda T$
For a M/G/1 model with a single server, if the input is the Poisson process, service time s is any general distribution (where $\bar{s}$ and $Var(s)$ both exist), then we have

$$N = \rho + \frac{a^2 + \lambda^2 Var(s)}{2(1-\rho)}, \quad (7)$$

which is the P-K formula for M/G/1 queuing model.

System Time of Queuing Model for Dynamic VRP

Dynamic VRP can be described as a single service vehicle traveling in a squared area A of length a at a constant velocity, v. For simplification of analysis, vehicle capacity is assumed to be adequate, i.e. the vehicle does not need to reload. Further, all demands are dynamic and demand locations are independently and uniformly distributed in the service area A. Inter-arrival time, u, is negative exponentially distributed with $\bar{u} = 1/\lambda$ and $Var(u) = 1/\lambda^2$. Let the on-site service time s' be general distributed where $\bar{s}'$ and $Var(s')$ both exist. Defined T_i as duration between arrival and completion of service for demand i, and W_i as duration between starting of service and arrival of service for demand i. Thus, we have $W_i = T_i - s_i$. The steady state system time $T = \lim_{i\to\infty} E[T_i]$, and the steady state waiting time $W = T - \bar{s}$. The objective of this study is to find a service policy that minimized T, denoted as T^*.

Based on the above definition, this dynamic VRP can now be treated as M/G/1 queuing model. Noting that, for a certain demand i, the service time s_i includes the on-site service duration s_i' and traveling time of d_i/v, duration between receiving demand request and traveling to demand position i, where d_i is the traveling distance and v is the traveling velocity. For a steady state system, we have

$$\lim_{n\to\infty} \tfrac{1}{n}\sum_{i=1}^{n} s_i = \lim_{n\to\infty} \tfrac{1}{n}\sum_{i=1}^{n} s_i' + \lim_{n\to\infty} \tfrac{1}{n}\sum_{i=1}^{n} \frac{d_i}{v},$$

that is:

$$\bar{s} = \bar{s}' + \frac{\bar{d}}{v}. \quad (8)$$

Assuming s_i' and d_i are not correlated, random variables s' and d are independently distributed, then,

$$Var(s) = Var(s') + Var\left(\tfrac{d}{v}\right). \quad (9)$$

Combining equation (7), (8), (9) and Little's formula, we have

$$T=\left(\bar{s}'+\frac{\bar{d}}{v}\right)+\frac{\lambda}{2}\frac{\left(\bar{s}'+\frac{\bar{d}}{v}\right)^2+Var(s')+\frac{Var(d)}{v^2}}{\left(1-\lambda\left(\bar{s}'+\frac{\bar{d}}{v}\right)\right)}. \tag{10}$$

MODELS OF REAL TIME POLICY FOR DYNAMIC VRP

First Come First Served Policy

In the First Come First Served (FCFS) policy, the vehicle services demands in the order of arrival. If the vehicle drops off a demand and there is no waiting demand. Hence, the vehicle awaits at the last demand's delivery location. Note that in this system, successive service times are not independent, and so any use of M/G/1 results is an approximation.

As demand locations are independently and uniformly distributed in service area A, then according to Equation (2) and (3), we have

$$\begin{aligned}\bar{d}&=E\left(\left|z_1\,z_2\right|\right)\approx 0.52a\\ Var\left(d\right)&=Var\left(\left|z_1z_2\right|\right)\approx 0.06a^2\end{aligned}. \tag{11}$$

Substitute Equation (11) into Equation (10), we will have the system time for FCFS policy.

$$T_{FCFS}=\left(\bar{s}'+\frac{0.52a}{v}\right)+\frac{\lambda}{2}\frac{\left(\bar{s}'+\frac{0.52a}{v}\right)^2+Var(s')+\frac{0.06a^2}{v^2}}{\left(1-\lambda\left(\bar{s}'+\frac{0.52a}{v}\right)\right)} \tag{12}$$

Median Reposition Policy

Differing from the static VRP, it is usually more important for dynamic VRP to minimize the waiting time for dynamic demand than to minimize the total distance traveled or make span. Bertsimas and Van Ryzin (1991) proposed a stochastic queue median policy to minimize the average system time. Based on this, we proposed a modification policy called median reposition policy. Thus, when considering there is no new demand waiting in line, the vehicle's waiting position can be repositioned in order to minimize demand waiting time.

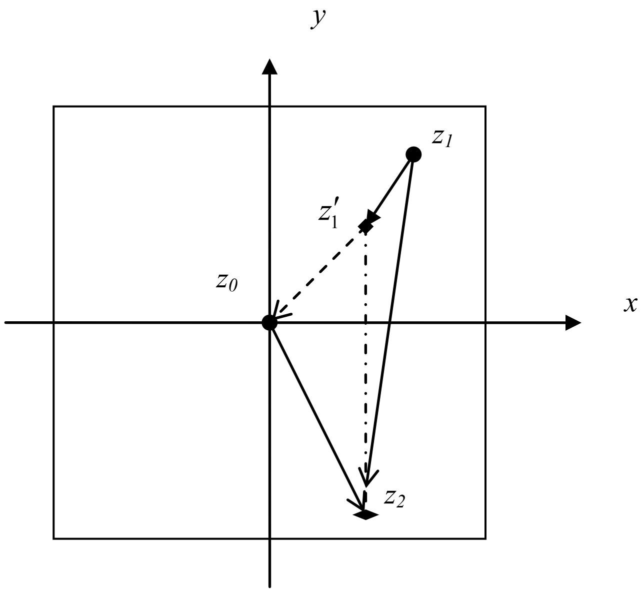
From Figure 1, we defined $\left|z_1z_2\right|=d$, and let $d'=\left|z_1'z_2\right|$. When the vehicle completes the service in z_1, if there is an immediate demand in z_2, the vehicle will travel from z_1 to z_2 directly. If there is no immediate demand in queue, the vehicle will then return to the median position z_0. When traveling to a position z_1', if a new demand in z_2 arrived, the vehicle will redirect to z_2 immediately. Otherwise, it will return to z_0 awaiting next order. In this case, we will have

$$T_{median}=\left(\bar{s}'+\frac{\bar{d}'}{v}\right)+\frac{\lambda}{2}\frac{\left(\bar{s}'+\frac{\bar{d}'}{v}\right)^2+Var(s')+\frac{Var(\bar{d}')}{v^2}}{\left(1-\lambda\left(\bar{s}'+\frac{\bar{d}'}{v}\right)\right)}. \tag{13}$$

As $E\left(\left|z_1' z_1\right|\right)=v\left(\frac{1}{\lambda}-T'\right)\geq v\left(\frac{1}{\lambda}-T\right)$, then

$$
\begin{aligned}
x_{z_1'} &= \frac{\left|z_1' z_0\right|}{\left|z_1 z_0\right|}x_{z_1} \approx \frac{0.38a - v\left(\frac{1}{\lambda}-T\right)}{0.38a}x_{z_1} = Cx_{z_1}\\
y_{z_1'} &= \frac{\left|z_1' z_0\right|}{\left|z_1 z_0\right|}y_{z_1} \approx \frac{0.38a - v\left(\frac{1}{\lambda}-T\right)}{0.38a}y_{z_1} = Cy_{z_1}\,. \qquad (14)\\
C &= \frac{0.38a - v\left(\frac{1}{\lambda}-T\right)}{0.38a}
\end{aligned}
$$

Figure 1: Median Reposition Policy for Dynamic VRP



This figure shows the graphical representation of the notation used. In which, z0 represent the median position, z1 is the prior service position and z2 is the immediate following demand position.

Follow the deduction of Equation (1) and (2), we have:

$$
\begin{aligned}
E\left((d')^2\right) &= \left(1+C^2\right)\frac{a^2}{6}\\
\bar{d}' &= \int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\left(\frac{C}{a}\right)^2\left(\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\frac{1}{a^2}\sqrt{\left(x_{z_2}-Cx_{z_1}\right)^2+\left(y_{z_2}-Cy_{z_1}\right)^2}\,dx_{z_2}\,dy_{z_2}\right)dx_{z_1}\,dy_{z_1}\,. \qquad (15)\\
Var\ (d') &= E\left((d')^2\right)-\left(\bar{d}'\right)^2
\end{aligned}
$$

THE SIMULATION EXAMPLE

Table 1 stated the system parameters for the example.

Table 1: Parameters for the Example

a= 1	*v =1*	*λ = 0.9*	S' follows [0,0.5] uniform Distribution $\bar{S}' = 0.25$ $Var(\bar{S}') \approx 0.02$

This table shows the parameters used for the simulation example. We simulate the models using length(a)=1, vehicle velocity(v)=1, the probability intensity(λ)=0.9, and an uniformly distributed s' of (0, 0.5).

From Equation (12), we can determine the system time for FCFS policy is $T_{FCFS} \approx 1.76$. From Equation (13), (14) and (15), we have $T_{median} \approx 1.65$.

With the system parameter in Table 1, we use Minitab software to generate simulated data, and from the generated data, we have:

$$\bar{T}_{FCFS} = \frac{1}{50}\sum_{i=1}^{50} T_{FCFS_i} \approx 1.74$$

$$\bar{T}_{median} = \frac{1}{50}\sum_{i=1}^{50} T_{median_i} \approx 1.60$$

Margin of error:

$$\frac{\left|\bar{T}_{FCFS} - T_{FCFS}\right|}{\bar{T}_{FCFS}} \times 100\% \approx 1.15\%$$

$$\frac{\left|\bar{T}_{median} - T_{median}\right|}{\bar{T}_{median}} \times 100\% \approx 3.12\%$$

The simulated result of $\bar{T}_{FCFS}$ and $\bar{T}_{median}$ is consistent with the theoretical result and the system time is reduced by $\frac{\left|\bar{T}_{median} - \bar{T}_{FCFS}\right|}{\bar{T}_{median}} \times 100\% \approx 8.05\%$ for a median reposition policy. It hence shows that median reposition policy is more effective in reducing the average customer waiting time than the FCFS policy.

CONCLUSION

Dynamic vehicle routing with previously unknown customer requests arriving over time has become increasingly important in the transportation industry as new technologies such as global positioning systems and wireless communications allow the assignment of new requests to vehicles in real time. In this paper, we have studied a dynamic VRP where new customer arrives at a uniformly chosen random location after the vehicles have left the depot. We examined the problem of choosing an appropriate service policy that mnimizes the waiting time and hence shortening the total makespan. System time for two real time strategies, FCFS and median repositioning, were calculated. With the simulated parameters, the obtained results clearly demonstrate the advantage of median reposition policy: Compared with the reference policy of FCFS, the median reposition policy was able to reduce the mean system time by approximately 8%.

There remain several avenues for future research. First, our paper focuses on the case of a single additional customer. A natural next step would be to explore further the problem variants of a nonuniform distribution of new customers and the arrival of more than one new customer. Further, our model is based on a single service vehicle. Some simplifications have been taken through the process of formula deduction, yet the result can be augmented into a more general situation, such as multiple vehicles with load limitations.

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