OPTION PORTFOLIO VALUE AT RISK USING MONTE CARLO SIMULATION UNDER A RISK NEUTRAL STOCHASTIC IMPLIED VOLATILITY MODEL

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ABSTRACT

This paper calculates option portfolio Value at Risk (VaR) using Monte Carlo simulation under a risk neutral stochastic implied volatility model. Compared to benchmark delta-normal method, the model produces more accurate results by taking into account nonlinearity, passage of time, non-normality and changing of implied volatility. Two parameters in the model: the correlation between underlying and the at –the-money implied volatility and the volatility of percentage change of the at- the-money implied volatility, can explain market skew phenomena quite well.

JEL: C63, G13, G17

KEYWORDS: Stochastic Implied Volatility Model, Value at Risk, Market Skew Phenomena

INTRODUCTION

The measurement of financial market risk is of primary importance for senior management and regulators. Value at risk (VaR) summarizes the worst loss of a portfolio over a given period with a given level of confidence (Jorion, 2000). VaR has become widely used by financial institutions, corporations and asset managers (Morgan 1996). The Basle Committee on Banking Supervision (BIS) and other central bank regulators also use VaR as a benchmark risk measure to determine the minimum amount of capital a bank is required to maintain as reserves against market risk (Pallota, Zenti 2000). There are some methods to calculate option portfolio VaR. The most widely used is the Delta normal method. Even though this method is simple and straightforward, it does not take into account option non-linearity, passage of time, changing implied volatility and non-normality of market price distribution (Hull and White 1998).

If the percentage of underlying price change were to follow a normal distribution, then the implied volatilities of all options with different strikes would be equal to each other, and if we draw a graph with implied volatilities as Y-axis, option strikes as X-axis, and then we would get a flat line. However, that is not case in the real world. In option market of equity and equity index instruments, we see a consistent left skew graph pattern. In commodity option markets though, a consistent right skew graph pattern shows up. This is widely known skew phenomena in option market. There are many papers that explain this phenomenon (Derman and Kani, 1994, Rubinstein, 1994, Hull and White, 1987, Heston, 1993, Stein and Stein 1991). Peng He and Stephen Yau followed a relative new stochastic implied volatility framework and made some modification in the model setup. This paper calculates Value at Risk under this model using Monte Carlo simulation, and compares the result with the benchmark, the Delta Normal Method. In addition, this paper examines how the model parameters explain skew phenomena.

The next section gives a detailed literature background about models used to explain skew phenomena, and a review of the stochastic implied volatility model developed by Peng He and Stephen Yau. The following section covers steps to do Monte Carlo simulation under this model, and the benchmark method, delta normal method. The case study and result section compares the simulation result with the benchmark in a case study of two option portfolios. In addition, this section explains market skew phenomena by using the two parameters in the model and shows the skew graph. The final session concludes.

LITERATURE REVIEW AND BACKGROUND

There is compelling evidence that exchange-traded options prices contain additional volatility information that can not backed out from the price information of the underlying security alone (He and Yau(2007), Christensen and Prabhala(1998), Cao and Chen (2000) etc). Therefore, instead of deriving prices for them, a pricing model should use their prices as input. Schonbucher (1998) and Ledoit and Santa Clara (1999) made this breakthrough and for the first time in the financial literature, the implied volatility is modeled as an input rather than as an output. Hafner (2004) presented a factor-based model of the stochastic evolution of the implied volatility surface. On the other hand, directly modeling the dynamics of implied volatilities is required by the nature of some exotic derivatives based on ATM (At-the-Money) implied volatility of an option written on a reference asset.

In previous paper (He and Yau (2006)), we developed a risk-neutral diffusion model for the stochastic market implied volatility. Unlike Hafner (2004) and Ledoit and Santa Clara (1998), we think modeling the whole implied volatility surface is dangerous because it is very difficult to guarantee no arbitrage between options with different strikes during the diffusion process of the corresponding implied volatilities of those options. Instead, we model one implied volatility only. Our model setup is also different from Schonbucher (1998). We think the percentage movement of implied volatility is better modeled than the implied volatility itself. The reason is the same as why people model the percentage movement of underlying asset price as opposed to underlying asset price itself. Furthermore, this modification can ensure implied volatility is positive during the diffusion process. After the proper setup, we derived the risk-neutral drift term of stochastic implied volatility, which is necessary to be no-arbitrage. We proved that the implied volatility of At-the-Money options mature immediately should converge to underlying volatility at rate of time to maturity, which specifies the stochastic process of underlying volatility. Finally, we developed model as follows:

$$dS_t = rS_t dt + \theta_t S_t dW_t \tag{1}$$

$$\begin{aligned} d\delta_t(\tau, X=1) &= (\frac{\delta_t^3}{24} - r\theta_t)dt + (r - \frac{\delta_t^2}{2})\beta\rho dt \\ &+ \delta_t \beta dZ + \partial_K \delta_t r S_t dt + \partial_K \delta_t \theta_t S_t dW_t \\ &+ \frac{1}{2}\partial_K^2 \delta_t \theta_t^2 S_t^2 dt + \theta_t S_t \beta \partial_K \delta_t \rho dt + \partial_T \delta_t dt \end{aligned} \tag{2}$$

$$\theta_t = \delta_t(\tau = 0, X = 1) \tag{3}$$

Where θ_t is instantaneous underlying volatility. $\delta_t(\tau, X)$ is the relative implied volatility indexed by time to maturity τ and moneyness $X = K / S_t$, the ratio between strike and underlying price. Therefore, $\delta_t(\tau = 0, X = 1)$ is the implied volatility of ATM option maturing immediately. ρ is the correlation coefficient between one Brownian motion Z and another Brownian motion W. β is the volatility of percentage change of implied volatility.

To simplify the model for use in practice, we assume $\partial_K \delta_t$, $\partial_K^2 \delta_t$ and $\partial_T \delta_t$ are zeros. The assumption is reasonable because empirical observations reveal that ATM implied volatilities are typically the same or change little for small strike (or underlying price) change and small maturity time change. This is in accordance with traders' "sticky delta rule."

METHODOLAGY

There are two approaches to compute VaR. The first approach is to use local valuation. Local valuation methods measure risk by valuing the portfolio once, at the initial time 0, and using local derivatives to deduce the possible movements. The second approach uses full valuation. Full valuation methods measure risk by fully re pricing the portfolio over a number of scenarios.

In the local valuation approach, practitioners calculate the VaR of option portfolios most commonly use the delta-normal method. It uses the linear, or delta derivatives and assumes normal distributions. The well-known formula is applied.

$$VaR(\alpha,\phi,T) = c_{N(0,1)}(\alpha)S_0 \mid D_\phi \mid \upsilon_{R_T} \tag{4}$$

Where $c_{N(0,1)}(\alpha)$ is the α -quantile of the standard normal distribution. S_0 is the initial underlying price. $\mid D_\phi \mid$ is the absolute delta of the option portfolio. υ_{R_T} is the volatility of return of the underlying during holding period $[0,T]$.

The full valuation approach uses Monte Carlo simulation or historical simulation to generate the probability distribution for portfolio value change ΔV_T. Let ΔV_T^j denote the change in portfolio value over $[0,T]$ in scenarios $j = 1,...J$. Then the distribution function $F_{\Delta V_T}(x)$ of ΔV_T is approximated by:

$$\hat{F}_{\Delta V_T}(x) = \sum_{\substack{j=1,...J \\ \Delta V_T^j \le x}} \frac{1}{J} \tag{5}$$

I completed Monte Carlo simulation on the model, as followings. Given a proper initial ATM implied volatility $\delta_0(0, X=1)$ or underlying volatility θ_t, underlying price S_0, interest rate r and two model parameters β, ρ, this dynamic system can be simulated from time 0 to time T as following:

1. Suppose at any time t, $0 < t \le T$, we have got $S_t, \theta_t(or \delta_t(t, X=1))$ and now we want to simulate $S_{t+\Delta t}$ and $\theta_{t+\Delta t}$ at time $t+\Delta t$, which $t+\Delta t \le T$ .The short time interval $\Delta t = (T-0)/N$.
2. At time t, generate the random next increment $\Delta W_t, \Delta Z_t$ of Brownian motion for use over the current time interval $[t, t+\Delta t]$. Since $\Delta W_t, \Delta Z_t$ are correlated, we set

$$\Delta W_t = \varepsilon_{1,t}\sqrt{\Delta t} \tag{6}$$

$$\Delta Z_t = \varepsilon_{1,t}\rho\sqrt{\Delta t} + \varepsilon_{2,t}\sqrt{1-\rho^2}\sqrt{\Delta t} \tag{7}$$

Where $\varepsilon_{1,t}, \varepsilon_{2,t}$ are two independent random numbers from standard normal distribution.

3. Approximate the solution of SDE equation for the underlying price by

$$\ln S_{t+\Delta t} = \ln S_t + (r - \frac{\theta_t^{\,2}}{2})\Delta t + \theta_t \Delta W_t \qquad (8)$$

The simulation equation for the underlying volatility is based on Euler approximation, and here it is:

$$\theta_{t+\Delta t} = \theta_t + (\frac{\theta_t^3}{24} - r\theta_t)\Delta t + (r - \frac{\theta_t^2}{2})\beta\rho\Delta t + \theta_t \beta dZ \qquad (9)$$

With $S_{t+\Delta t}$, $\theta_{t+\Delta t}$, now start to compute parameters at time $t+\Delta t$, return to Step 1, reset t to $t+\Delta t$ and iterate until time T generate trajectories of S_t and θ_t .

RESULTS

The paper provides a case study to show difference between the delta-normal method and Monte Carlo simulation approaches. Consider two option portfolios. Suppose the underlying price is 100 and risk free interest rate is zero. The two portfolios consist of the following instruments. 1. Portfolio 1 (PF1): A long position in 100 call option with strike $K = 100$ and maturity date is 42 days away and 2. Portfolio 2 (PF2): A long position in 100 ATM straddles (long position in both call and put option with same strike and same maturity) with strike $K = 100$ and maturity date is 42 days away.

At time zero, the portfolio values $V_0^{PF_i}, i = 1,2,$ are:

$$\begin{aligned} V_0^{PF_1} &= 100 * C_0(S_0 = 100, K = 100, r = 0, \\ \sigma &= 0.4, 42/365) = 541.90 \end{aligned} \qquad (10)$$

$$\begin{aligned} V_0^{PF_2} &= 100 * (C_0(S_0 = 100, K = 100, \\ r &= 0, \sigma = 0.4, 42/365) \\ &+ P_0(S0 = 100, K = 100, r = 0, \\ \sigma &= 0.4, 42/365)) = 1081.80 \end{aligned} \qquad (11)$$

The objective of the following analysis is to compute the VaR for both portfolios over one business and ten business days. The confidence level is set to 95%. First consider the delta normal method. The Black Scholes deltas for the above call and put option are 0.5270, -0.4730 respectively, so the portfolio deltas are:

$$D_{PF_1} = \frac{\partial V_0^{PF_1}}{\partial S} = 100 * 0.5270 = 52.70 \qquad (12)$$

$$D_{PF_2} = \frac{\partial V_0^{PF_2}}{\partial S} = 100 * (0.5270 - 0.4730) = 5.41 \qquad (13)$$

Using $C_{N(0,1)}(5\%) = \ 1.645$, and volatility 0.4, the VaR for each portfolio and each holding period is computed according to the formulas. Table 1 column, Delta-normal, lists the result.

Now consider Monte Carlo simulation. The parameters, chosen are $\theta_0 = 0.4, \rho = -0.5, \beta = 1$. Following the procedure described as above, we generate 10,000 scenarios of S, σ over the time horizon $[0, T]$, where T equals 1 business day or 10 business days, respectively. For each scenario, we compute the portfolio value and the change in portfolio value. This provides us with a simulated distribution function $\hat{F}_{\Delta V_T}(x)$. Given $\hat{F}_{\Delta V_T}(x)$, VaR is easily computed as the negative $\alpha-$quantile of ΔV_T. Table 1 column "Paper Monte Carlo" lists the result.

As shown in Table 1, the VaR computed under delta normal method is quite different from that computed under the paper Monte Carlo simulation. For portfolio 2, under the delta-normal method, the VaR is lower than the VaR under paper Monte Carlo simulation, whereas for portfolio 1 it is opposite. This can be explained as follows: Since the straddle position is almost delta neutral, the VaR is very small under the delta-normal method. This, however, indicates one disadvantage of the delta-normal method. The delta-normal method is a linear approximation, which cannot capture the nonlinearity of options. In this case, the straddle position is exposed to relative big Gamma risk and Gamma is the quadratic part. On the other hand, considering portfolio 1, the delta-normal method produces higher VaR than it should. The obvious example is that 10-day holding period VaR for portfolio 1 under Delta normal method is 694 Dollar. The maximum amount of money one can lose when holding that call option, Portfolio 1 over 10 days (or any day) is the initial option premium, i.e. 541.90 Dollar. The reason the delta normal method produces too high a VaR for portfolio 1 is that the delta normal method does not account for the time decay of option prices.

In fact, besides nonlinearity and passage of time, there are two other effects, non-normality and implied volatility variations, which the delta normal method fails to catch. The delta normal method assumes the underlying return R_T is normally distributed. However convincing empirical studies have shown that underlying return tend to exhibit fat-tailed distributions. In other words, extremely low and high returns have greater probability than assigned by the normal distribution. The kurtosis of the log Terminal Price distribution for the above parameter set is 4.06, which explains fat-tailed distribution. In addition, the delta normal method assumes a constant underlying volatility, and thus constant implied volatilities. In fact, implied volatilities change over time. In contrast, the paper Monte Carlo simulation captures all the effects mentioned in the above. It is a much better method to compute VaR. The only disadvantage of the paper Monte Carlo simulation is computation time. However using today's more and more powerful computers, the computation runs increasingly fast.

Table1 Delta-normal VaR and Paper Monte Carlo simulation VaR for a confidence level of 95%

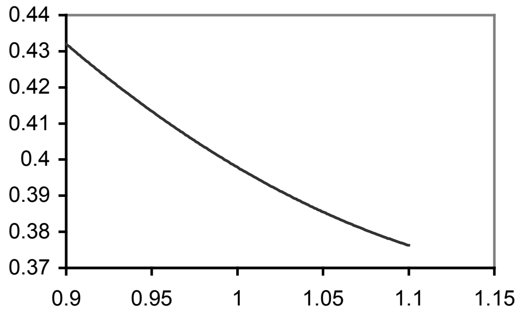
	Delta-normal		**Paper Monte Carlo**	
Holding Period	1 Day VaR	10 Day VaR	1 Day VaR	10 Day VaR
Portfolio 1	219	694	207	497
Portfolio 2	23	71	109	373

This table shows the delta-normal VaR and Paper Monte Carlo Simulation. Results are in U.S. Dollars.

Next we study how VaR changes with parameters in the paper model, namely β and ρ. Those two parameters can explain market skew phenomena quite well. For zero correlation between the underlying and volatility $(\rho = 0)$, the implied volatility curve is symmetric and skewness is around zero. Smile-shape volatility curves are commonly observed for options on a foreign currency as shown in Figure 1. Our analysis is consistent with the empirical study (Bates 1996), which show the correlation between implied volatilities and the exchange rate is close to zero. For negative correlation $(\rho < 0)$, the implied volatility curve skews to left and skewness is negative. Skew-shape volatility curves are commonly observed for options on equities and equity indices. Our analysis is consistent with the empirical study (Christie 1982), which show the volatility of an equity price tend to be negatively correlated with the equity price. For positive correlation $(\rho > 0)$, the implied volatility curve skews to right and the skewness is positive. The bigger the absolute value of ρ, the more skewed of the curve and the bigger the absolute value of the skewness. On the other hand, the volatility of percentage change of implied volatility β has an effect on the curvature of implied volatility curves. With other parameter held fixed, the larger the β, the larger the curvature of the implied volatility curve, the larger the kurtosis.

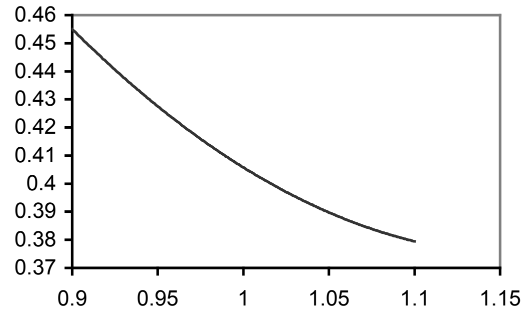
Figure1 Implied volatility Curve and The moments of Log Terminal Underlying Price Distribution

$\rho = -0.5, \beta = 1$



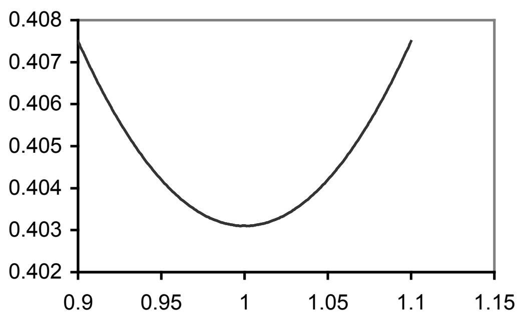
Distribution: Skewness=-0.053 Kurtosis = 4.027

$\rho = -0.5, \beta = 2$



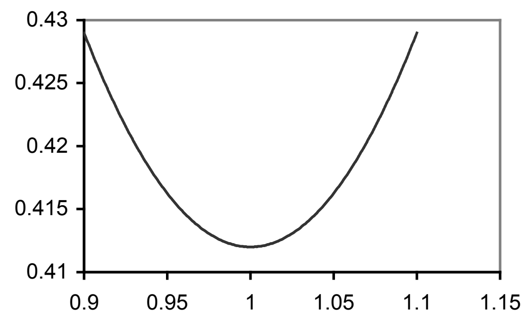
Distribution: Skewness = -0.41 Kurtosis = 10.6

$\rho = 0, \beta = 1$



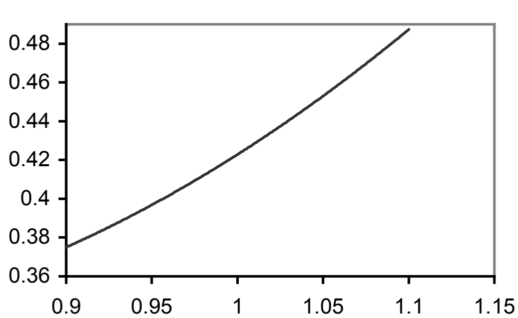
Distribution: Skewness=-0.012 Kurtosis = 3.54

$\rho = 0, \beta = 2$



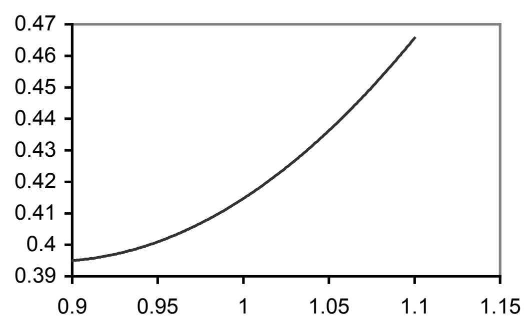
Distribution: Skewness= -0.066 Kurtosis = 7.055

$\rho = 0.5, \beta = 1$



Distribution: Skewness = 0.057 Kurtosis = 3.836

$\rho = 0.5, \beta = 2$



Distribution: Skewness = 0.385 Kurtosis = 10.728

Note: Simulated Implied Volatility Curves for options with maturity of 42 days. The initial underlying volatility θ_0 *is 0.4 and interest rate* r *is zero. The moments (Skewness and Kurtosis) are the moments of log terminal underlying price distribution.*

VaR of the two option portfolios for different β and ρ are calculated and the results are presented in the Table 2. The higher the skewness and kurtosis of the log terminal underlying distribution $(\ln S_T)$, the higher VaR value of both option portfolios.

Table 2 Paper Monte Carlo simulation VaR of two option portfolios for different β and ρ for a confidence level of 95%

				Portfolio 1		Portfolio 2	
β	ρ	Skewness	Kurtosis	1 Day VaR	10 Day VaR	1 Day VaR	10 Day VaR
1	-0.5	-0.053	4.027	207	497	109	373
1	0	-0.012	3.54	206	492	113	393
1	0.5	0,057	3.836	206	497	109	369
2	-0.5	-0.409	10.578	224	514	203	537
2	0	-0.066	7.055	225	512	213	573
2	0.5	0.385	10.728	226	513	202	546

This table shows paper monte carlo simulation VaR of two option portfolios. Results are in U.S. Dollars.

CONCLUDING COMMENTS

There are some limitations in the benchmark method for option portfolio VaR calculation. In a related note, skew phenomena are widely known in the option market and there are many research papers to explain this. This paper calculates VaR under a stochastic implied volatility using Monte Carlo Simulation, and compares the results with the benchmark, Delta normal method.

The VaR calculation is more accurate because it considers nonlinearity, passage of time, non-normality, changing of implied volatility. In addition, two parameters in the model, the correlation between underlying and the at-the-money implied volatility and the volatility of percentage change of the at-the-money implied volatility, can explain market skew phenomena quite well. The ultimate goal of this study is to develop a method, which we can use in the real world. Therefore, the limitation of this paper comes from two aspects. The first aspect is that we need develop a simple way to calibrate the model using real market data. In this paper, when I do Monte Carlo simulation, I assume parameters, θ_0, ρ, β. It would be better if we can retrieve those parameters by calibrating to model to real market data. The second aspect is that the computation of VaR can somehow be simpler and quicker, than the benchmark method. Therefore, future studies can either somehow increase the calculation speed or somehow find an approximate analytical solution to the model, so that the computation would be quicker.

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