THE HESTON STOCHASTIC VOLATILITY MODEL FOR SINGLE ASSETS AND FOR ASSET PORTFOLIOS: PARAMETER ESTIMATION AND AN APPLICATION TO THE ITALIAN FINANCIAL MARKET

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ABSTRACT

We investigate the performance of the Heston stochastic volatility model in describing the probability distribution of returns both in the case of single assets and in the case of asset portfolios. The R. parameters of the Heston model are estimated from observed market prices using a simple calibration method based on an integral representation of the exact probability density function of returns derived by Dragulescu and Yakovenko (2002). In the case of multiple correlated assets, the correlation parameters are obtained using a heuristic procedure based on a matrix completion algorithm. We present numerical experiments where several stocks traded on the Italian financial market are considered. We show that, both in the case of single assets and in the case of multiple correlated assets, the Heston model provides an excellent agreement with historical time series data and fits the empirical probability distribution of returns far better than the lognormal model.

INTRODUCTION

In this paper we assess the performances of the Heston model (HM) in describing the probability distribution of stock returns on the Italian financial market. At the same time we propose a simple method to calibrate the HM that gives an excellent agreement with historical time-series data both in the case of single and multiple correlated assets.

The paper is organized as follows: in Section 2 we provide a review of related literature. In Section 3 we briefly recall the basic facts about the LM. In Section 4 we give a description of the HM in the case of single and multiple correlated assets. In Section 5 we describe the calibration method used to estimate the parameters of the HM. Finally, in Section 6 we present and discuss the numerical results obtained applying the calibration algorithm developed in Section 5.

LITERATURE REVIEW

The dynamics of stock market prices is often described by the lognormal model (LM). Based on the assumption of constant drift and volatility, the LM gives a normal probability distribution of asset returns and therefore it is a very simple and tractable model. This is the reason why the LM, originally introduced by Bachelier (1900) and refined by Osborne (1959), is still nowadays very popular among financial researchers and practitioners.

Nevertheless many empirical studies on financial markets show that the probability distribution of stock returns is far from being normal. In particular, empirical observations of option prices reveal that the volatility of the underlying stocks varies as a function of the strike prices (the so-called smile effect, see Wilmott, 1998). Moreover the probability distribution of realized returns is often leptokurtic, i.e. it has fatter tails and higher peaks than the lognormal probability distribution (Bouchaud & Potters, 2001, Fama, 1965).

This empirical evidence motivated several authors to reject the assumption of constant volatility and to introduce the so-called stochastic volatility models; that is, models where the asset price volatility is described as a stochastic process. Among the stochastic volatility models that can be found in the literature, see for instance Heston, 1993, Hull & White, 1987, Melino & Turnbull, 1990, Scott, 1987, Stein & Stein, 1991, the Heston model (Heston, 1993) has received considerable attention since it gives an adequate description of stock market dynamics (Dragulescu & Yakovenko, 2002, Prange, Silva & Yakovenko, 2004) and yields tractable closed-form solutions (Dragulescu & Yakovenko, 2002, Heston, 1993).

In the financial literature the calibration of the HM is usually performed using two different approaches: in Aboura, 2004, Forbes, Martin & Wright 2002, Heston, 1993, Pan, 2002, the HM is calibrated consistently with observed option prices, while in Daniel, Joseph & Bree, 2005, Dragulescu & Yakovenko, 2002, Prange, Silva & Yakovenko 2004, Silva & Yakovenko, 2001 the parameters of the HM are estimated by fitting the probability distribution of realized asset prices.

In this paper we follow the latter approach since several stocks traded on the Italian financial market do not have options written on them. In particular, in the case of single assets, the calibration method proposed in this manuscript is similar to the one developed by Dragulescu and Yakovenko (2002). In Dragulescu & Yakovenko, 2002, an integral representation of the probability density function of returns of the HM is derived (Formula (23) p. 446). Note that this formula is an exact formula that gives the probability distribution of returns conditioned to the value taken by the initial variance. In addition Dragulescu and Yakovenko (2002) obtain also another expression for the probability density function of returns of the HM (Formula (28) p. 446), where the initial variance does not appear. This second formula is an approximate formula, since it is based on the assumption that the probability distribution of the initial variance is equal to the steady state probability distribution of the variance process.

The approximate formula of Dragulescu & Yakovenko (2002) is used in Daniel, Joseph & Bree, 2005 to calibrate the HM against the Dow Jones Industrial Average, S&P 500, FTSE 100 indexes, in Dragulescu & Yakovenko, 2002 to estimate the parameters of the HM against Dow Jones index time-series data, in Prange, Silva & Yakovenko, 2004 to estimate the parameters of the HM for several stocks belonging to the Dow Jones index, and in Silva & Yakovenko, 2001 to calibrate the HM for the S&P 500, NASDAQ and Dow Jones indexes. Note that in these works the value taken by the initial variance of the asset returns is not estimated, since it is not contained in the approximate formula of Dragulescu & Yakovenko, 2002.

The calibration method developed in this paper is based on the exact closed-form expression of the probability density function of the HM derived in Dragulescu & Yakovenko, 2002, Formula (23) p. 446. In particular we treat the initial variance of asset returns as an additional parameter of the model, so that we do not have to assume that the initial variance has stationary probability distribution, as done in Daniel, Joseph & Bree, 2005, Dragulescu & Yakovenko, 2002, Prange, Silva & Yakovenko, 2004, Silva & Yakovenko, 2001. Moreover, contrary to the calibration methods proposed in Daniel, Joseph & Bree, 2005, Dragulescu & Yakovenko, 2002, Prange, Silva & Yakovenko, 2004, Silva & Yakovenko, 2001, our approach allows to estimate the value taken by the initial variance.

First of all the method to calibrate the HM is developed in the case of single assets and then it is extended to the case of multiple correlated assets. In this latter case the correlation parameters of the HM are estimated using a heuristic technique based on a suitable matrix completion algorithm.
We estimate the parameters of the HM for several stocks belonging to the Italian Stock Exchange using historical data on a daily basis from June 2002 to June 2006. The results obtained show that using the calibration algorithm proposed in this paper the HM provides an excellent agreement with empirical data

both in the case of single and multiple correlated assets. In particular the HM captures the kurtosis effect exhibited by the empirical probability distribution of returns.

We point out that the contribution of this paper is twofold. First, we show that the HM describes the probability distribution of returns far better than the LM for stocks belonging to the Italian financial market. Second, we propose a method to calibrate the HM that is very easy to implement and performs very well both in the case of single assets and in the case of asset portfolios. From the practical standpoint we believe that these results can be very interesting for a stock trader. In fact it is crucial for a financial investor to use a mathematical model of stock prices which is simple to calibrate and provides a good agreement with realized returns.

THE MODEL

The Lognormal Model

Let $S(t)$ denote the price of an asset at time t and let t_0 denote the current time. According to the lognormal model (LM), $S(t)$ is described as a stochastic process satisfying the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma dW(t), \quad t \geq t_0, \tag{1}$$

with initial condition:

$$S(t_0) = S_0. \tag{2}$$

In (1), μ and σ are constant parameters, called drift and volatility, respectively, and $W(t)$ is a standard Wiener process.
Let us define the asset return over the time interval $[t_0, t]$:

$$X(t) = \log\left(\frac{S(t)}{S_0}\right), \quad t \geq t_0. \tag{3}$$

Using Ito's lemma, equation (1) and initial condition (2) can be rewritten as follows:

$$dX(t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW(t), \quad t \geq t_0, \tag{4}$$

$$X(t_0) = 0. \tag{5}$$

The parameters μ and σ can be estimated from realized asset prices in a very simple way. In fact let us consider a set of equally spaced time values t_0, t_1, …,t_n and let us define $\Delta t = t_k - t_{k-1}$, $k=1,2,\ldots,n$. Equation(4) implies that asset returns are distributed as follows:

$$\log\left(\frac{S(t_k)}{S(t_{k-1})}\right) \approx Normal\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma\sqrt{\Delta t}\right), \; k = 1,2,\ldots,n, \tag{6}$$

where $Normal(a,d)$ denotes the normal probability distribution with mean a and standard deviation d.

Let S_k denote the asset price observed at time t_k, k=1,2,…,n, let us consider the realized returns:

$$x_k = \log\left(\frac{S_k}{S_{k-1}}\right),\ k = 1,2,\ldots,n, \tag{7}$$

and let *mean* and *var* respectively denote the sample mean and the sample variance of the realized returns:

$$mean = \frac{1}{n}\sum_{k=1}^{n} x_k, \quad var = \frac{1}{n-1}\sum_{k=1}^{n}(x_k - mean)^2. \tag{8}$$

The probability distribution of returns (6) yields the following statistical estimators for the parameters σ and μ:

$$\sigma = \sqrt{\frac{var}{\Delta t}}, \quad \mu = \frac{mean}{\Delta t} + \frac{\sigma^2}{2}. \tag{9}$$

Now let us consider the LM in the case of multiple correlated assets. Let m denote the number of assets considered and let $S_k(t)$ denote the price of the i-th asset at time t, i=1,2,…,m. Let us define the asset returns:

$$X_i(t) = \log\left(\frac{S_i(t)}{S_i(t_0)}\right),\ t \geq t_0,\ i = 1,2,\ldots,m, \tag{10}$$

equation (4) and initial condition (5) are generalized to the case of m correlated assets as follows:

$$dX_i(t) = \left(\mu_i - \frac{\sigma_i^2}{2}\right)dt + \sigma_i dW_i(t),\ t \geq t_0,\ i = 1,2,\ldots,m, \tag{11}$$

$$X_i(t_0) = 0,\ i = 1,2,\ldots,m, \tag{12}$$

where μ_i and σ_i represent the drift and the volatility of the i-th asset, respectively, and $W_i(t)$ is a standard Wiener process, i=1,2,…,m. The correlation coefficient between $W_i(t)$ and $W_j(t)$, which we denote with $\rho_{i,j}$, is assumed to be constant, i=1,2,…,m, j=1,2,…,m.

According to equation (11) the vector of stochastic variables [$X_1(t)$, $X_2(t)$, …, $X_m(t)$] has a multivariate normal distribution. Therefore in analogy with the case of single assets the parameters μ_i, σ_i, $\rho_{i,j}$, i=1,2,…,m, j=1,2,…,m, can be estimated from observed asset prices as follows.
Let $x_{i,k}$ denote the realized return of the i-th asset over the time interval [t_{k-1}, t_k], i=1,2,…,m, k=1,2,…,n. Let us define:

$$mean_i = \frac{1}{n}\sum_{k=1}^{n} x_{i,k}, \quad var_i = \frac{1}{n-1}\sum_{k=1}^{n}(x_{i,k} - mean_i)^2,\ i = 1,2,\ldots,m, \tag{13}$$

$$cov_{i,j} = \frac{1}{n-1}\sum_{k=1}^{n}(x_{i,k} - mean_i)(x_{j,k} - mean_j),\ i = 1,2,\ldots,m, \quad j = 1,2,\ldots,m. \tag{14}$$

The parameters σ_i, μ_i, $\rho_{i,j}$, i=1,2,…,m, j=1,2,…,m, can be estimated as follows:

$$\sigma_i = \sqrt{\frac{var_i}{\Delta t}}, \quad \mu_i = \frac{mean_i}{\Delta t} + \frac{\sigma_i^2}{2}, \quad i = 1,2,...,m, \tag{15}$$

$$\rho_{i,j} = \frac{cov_{i,j}}{\sigma_i \sigma_j}, \quad i = 1,2,...,m, \quad j = 1,2,...,m. \tag{16}$$

The Heston Model

Let us consider the HM in the case of single assets. According to the HM the stock price volatility is no longer assumed to be constant. Therefore equations (4)-(5) are rewritten as follows:

$$dX(t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma(t)dW(t), \quad t \geq t_0 \tag{17}$$

$$X(t_0) = 0. \tag{18}$$

Let us define the variance $V(t)$:

$$V(t) = \sigma^2(t), \quad t \geq t_0, \tag{19}$$

The variance $V(t)$ is modelled as a stochastic process satisfying the stochastic differential equation:

$$dV(t) = \gamma(\vartheta - V(t)) + \kappa\sqrt{V(t)}dW^{(1)}(t), \quad t \geq t_0, \tag{20}$$

with initial condition:

$$V(t_0) = v_0. \tag{21}$$

In (20) $\vartheta, \gamma, \kappa$ are positive constant parameters, and $W^{(1)}(t)$ is a standard Wiener process. Let η denote the correlation coefficient between $W(t)$ and $W^{(1)}(t)$, η is assumed to be constant.

The stochastic differential equations (17), (20) with initial conditions (18), (21) constitute the Heston stochastic volatility model for a single asset (Heston, 1993). These equations can be generalized to the case of m assets as follows:

$$dX_i(t) = \left(\mu_i - \frac{V_i(t)}{2}\right)dt + \sqrt{V_i(t)}dW_i(t), \quad t \geq t_0,\ i = 1,2,\ldots,m, \tag{22}$$

$$dV_i(t) = \gamma_i\big(\vartheta_i - V_i(t)\big)dt + \kappa_i\sqrt{V_i(t)}dW_i^{(1)}(t), \quad t \geq t_0,\ i = 1,2,\ldots,m, \tag{23}$$

$$X_i(t_0) = 0,\ i = 1,2,\dots,m, \tag{24}$$

$$V_i(t_0) = v_{0i},\ i = 1,2,\dots,m. \tag{25}$$

In (22), (23) the standard Wiener processes $W_i(t)$ and $W_i^{(1)}(t)$, i=1,2,…,m, j=1,2,…,m, are assumed to be correlated by a constant correlation matrix. In particular let Λ denote the correlation matrix of the 2m-dimensional Wiener process [$W_1(t)$, $W_2(t)$, …, $W_m(t)$, $W_1^{(1)}(t)$, $W_2^{(1)}(t)$,…, $W_m^{(1)}(t)$], we can represent Λ as the following block matrix:

$$\Lambda = \begin{bmatrix} \Sigma & E \\ E^T & B \end{bmatrix}, \tag{26}$$

where Σ, B are symmetric $m \times m$ matrices and E is a $m \times m$ matrix. We denote with $\rho_{i,j}$ the correlation coefficient between W_i and W_j, with $\beta_{i,j}$ the correlation coefficient between $W_i^{(1)}$ and $W_j^{(1)}$, and with $\eta_{i,j}$ the correlation coefficient between W_i and $W_j^{(1)}$, i=1,2,…,m, j=1,2,…,m. We have:

$$\Lambda = \begin{bmatrix} 1 & \cdots & \rho_{1,m} & \eta_{1,1} & \cdots & \eta_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{1,m} & \cdots & 1 & \eta_{m,1} & \cdots & \eta_{m,m} \\ \eta_{1,1} & \cdots & \eta_{m,1} & 1 & \cdots & \beta_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_{1,m} & \cdots & \eta_{m,m} & \beta_{1,m} & \cdots & 1 \end{bmatrix} \tag{27}$$

Estimation of the Parameters of the Heston Model

First of all we describe the calibration method used to estimate the parameters of the HM in the case of single assets, i.e. we consider equations (17)-(21).
Let $\widetilde{p}(t - t_0, x \mid v_0)$ denote the probability density function of having $X(t)$=x given $X(t_0)$=0 and $V(t_0)$=v_0. In Dragulescu & Yakovenko, 2002 the following integral representation of $\widetilde{p}(t - t_0, x \mid v_0)$ is derived:

$$\widetilde{p}(t - t_0, x \mid v_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\Phi(\omega, t - t_0, x, v_0)} d\omega, \tag{28}$$

where

$$\Phi(\omega, t - t_0, x, v_0) = i\omega(x + \mu(t - t_0)) - v_0 \frac{2(\omega^2 - i\omega)}{2\Gamma + \Omega \coth(\Omega)(t - t_0)} - \frac{2\gamma\vartheta}{\kappa^2} \cdot$$

$$\log\left(\cosh \tfrac{\Omega(t - t_0)}{2} + \frac{\Gamma}{\Omega} \sinh \tfrac{\Omega(t - t_0)}{2}\right) + \frac{\gamma\, \Gamma\, \vartheta(t - t_0)}{\kappa^2}, \tag{29}$$

$$\Gamma = \gamma + i\eta\kappa\omega, \tag{30}$$

$$\Omega = \sqrt{\Gamma^2 + \kappa^2(\omega^2 - i\omega)}. \tag{31}$$

Let us consider a set of equally spaced time values t_0, t_1, …,t_n, and let us define $\Delta t = t_k - t_{k-1}$, k=1,2,…,n. Let x_k denote the realized return of a given asset over the time interval $[t_{k-1},t_k]$, k=1,2,…,n. We estimate the parameters $\vartheta, \gamma, \kappa, \eta, \mu$,and the initial variance v_0 using the procedure outlined below:

1. We obtain the empirical probability density function of returns, which we denote with $p_{emp}(\Delta t, x)$, as follows. Let us consider the set of realized returns $\{x_1, x_2, \ldots, x_n\}$, and let x_{min} and x_{max} denote the minimum and the maximum of the set $\{x_1, x_2, \ldots, x_n\}$ respectively. We divide the interval $[x_{min}$, $x_{max}]$ into N bins of equal size. Let Δx denote the size of each bin, and let $\bar{x}_k$ denote the center of the k-th bin, k=1,2,…, N. At the centers of the bins, we evaluate the empirical probability density function of returns as follows:

$$p_{emp}(\Delta t, \bar{x}_k) = \frac{n_k}{n}, \quad k = 1,2,\ldots, N. \tag{32}$$

2. We treat the initial variance v_0 as a parameter of the HM. More precisely let $p(\Delta t, x)$ denote the probability density function of having $X(t + \Delta t) = x$ given $X(t) = 0$. Instead of formula (28), we consider the following one:

$$p(\Delta t, x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\Phi(\omega, \Delta t, x, v_0)} d\omega, \tag{33}$$

that is we assume that the right hand side of equation (28) represents the probability density function of having $X(t + \Delta t) = x$ given $X(t) = 0$ at every time $t \geq t_0$ (and not only at time $t = t_0$ when the variance takes the value v_0).

3. We estimate $\vartheta, \gamma, \kappa, \eta, \mu$,v_0 by minimizing the mean-square deviation, *msd*, between the functions defined in (32) and (33) evaluated at the centers of the bins:

$$msd = \sum_{k=1}^{N}\left[p_{emp}(\Delta t, \bar{x}_k) - p(\Delta t, \bar{x}_k)\right]^2. \tag{34}$$

Note that in order to compute $p(\Delta t, \bar{x}_k)$ the evaluation of the integral appearing in formula (33) is required. In our numerical experience, a fast and accurate numerical approximation of this integral can be computed using Simpson's quadrature rule.

Now let us consider the case of a portfolio of m correlated assets, i.e. equations (22)-(25). We note that the calibration algorithm described above, applied to every single asset of the portfolio, allows to estimate the parameters $\vartheta_i, \gamma_i, \kappa_i, \eta_{i,i}, \mu_i, v_{0i}$, i=1,2,…,m. Therefore the parameters of the HM that still need to be determined are the following elements of the correlation matrix (27): $\rho_{i,j}, \beta_{i,j}, i = 1,2,\ldots,m,\ j = 1,2,\ldots,m,$ and $\eta_{i,j}, i = 1,2,\ldots,m,\ j = 1,2,\ldots,m, i \neq j.$

The estimation of these parameters is a very difficult task. In fact, in the case of multiple correlated assets, neither an exact nor an approximate formula for the joint probability distribution of returns of the HM is available in the literature. As a consequence the correlation matrix Λ cannot be estimated by fitting the empirical probability distribution of portfolio returns with some analytical law. Therefore we resort to a heuristic procedure, whose description is given below.

The correlation parameters $\rho_{i,j}, i=1,2,\ldots,m, j=1,2,\ldots,m,$ are evaluated using relations (16), that is the correlations among asset prices are determined as if asset returns were normally distributed. Finally the parameters $\beta_{i,j}, i=1,2,\ldots,m, j=1,2,\ldots,m,$ and $\eta_{i,j}, i=1,2,\ldots,m, \ j=1,2,\ldots,m, \ \ i\neq j,$ are obtained using a matrix completion method that exploits the particular block structure of the matrix (27). A description of this algorithm is presented below.
Let us consider the Cholesky decomposition of the matrix Λ:

$$\Lambda = CC^T, \tag{35}$$

where C is a $2m\times 2m$ lower triangular matrix. Let us denote with $c_{i,j}$ the element of the i-th row and j-th column of the matrix C, $i=1,2,\ldots,2m,$ $j=1,2,\ldots,2m.$ Since Λ has the block structure (26) the elements belonging to the first m rows and the first m columns of matrix C are obtained by Cholesky decomposition of matrix Σ.

The elements of matrix C that still need to be determined, namely $c_{i,k}$, $i=m+1,m+2,\ldots,2m,$ $k=1,2,\ldots,i,$ are obtained using the following relations:

$$c_{i,k} = \begin{cases} \dfrac{\eta_{i-m,i-m}}{c_{i-m,i-m}} \ \text{if} \ \left|\eta_{i-m,i-m}\right| < \left|c_{i-m,i-m}\right|, \quad i=m+1,m+2,\ldots,2m, \ \ k=i-m, & \text{(38a)} \\ 0 \ \text{if} \ \left|\eta_{i-m,i-m}\right| < \left|c_{i-m,i-m}\right|, \quad i=m+1,m+2,\ldots,2m, \ \ k=1,2,\ldots,i-1, k\neq i-m, & \text{(38b)} \\ \eta_{i-m,i-m}c_{i-m,k} \ \text{if} \ \left|\eta_{i-m,i-m}\right| \geq \left|c_{i-m,i-m}\right|, \quad i=m+1,m+2,\ldots,2m, \ \ k=1,2,\ldots,i-1, & \text{(38c)} \\ \sqrt{1-\sum_{j=1}^{i-1} c_{i,j}^2}, \quad i=m+1,m+2,\ldots,2m, \ \ k=i. & \text{(38d)} \end{cases}$$

It can be easily checked that, thanks to equations (38a)-(38c) the correlation coefficient between the return and the variance of the i-th asset is equal to $\eta_{i,i}$, $i=1,2,\ldots,m.$ Note also that Equations (38d) ensure that $\beta_{i,i}=1$, $i=1,2,\ldots,m,$ so that the matrix CC^T is a positive definite symmetric matrix with unitary diagonal elements, that is a valid correlation matrix. It is important to observe that, according to equations (38a)-(38c), if $|\eta_{i-m,i-m}|<|c_{i-m,i-m}|$, that is the magnitude of the correlation between the price and the volatility of the i-th asset is smaller than $|c_{i-m,i-m}|$, the i-th row of matrix C has at most two nonzero elements, $i=m+1,m+2,\ldots,2m.$ Moreover we note that if the correlation between the price and the volatility of a given asset is zero, then we would reasonably expect that the correlations of the volatility of that asset with the prices and the volatilities of the other assets are zero as well. This property is respected by the matrix completion algorithm (38a)-(38d). In fact, given an integer i, $i=1,2,\ldots,2m,$ if

$\eta_{i,i} = 0$ from equations (38a), (38b) we obtain $c_{m+i,j} = 0$, $j = 1,2,\ldots,m+i-1,$ and hence from relations (27), (35) we have $\eta_{i,k} = 0$, $k = 1,2,\ldots,m,$ and $\beta_{i,k} = 0$, $k = 1,2,\ldots,m,$ $k \neq i.$

Data

We estimate the parameters of the HM for the following six stocks belonging to the Italian Stock Exchange: Autostrade SpA, Mediobanca SpA, Pirelli & C SpA, RaS Holding, Unicredito Italiano SpA, Snam Rete Gas.

The set of historical data used to derive the empirical probability distribution of returns (32) consists of daily observed asset prices from 17 June 2002 to 15 June 2006. Note that we consider only historical data starting from June 2002 since we want to exclude from our analysis the crash of the Italian financial market due to September 11, 2001. In fact in order to take into account the effects of extreme and unpredictable events such as the the September 11, 2001 terroristic attacks, one should use more ad-hoc models of stock price dynamics, e.g. models with jumps in returns and in volatility (see Eraker, Johannes & Polson, 2003 and references therein). On the other hand we observe that for some of the stocks considered, the historical data are not available on time periods significantly longer than four years (for instance the Snam Rete Gas stock was not quoted on the Italian market before December 2001). We consider only returns on a daily basis, since lower observation frequencies are incompatible with a set of historical data spanning a time horizon of only four years. In fact we have found that the empirical probability distributions of returns computed using time lags longer than 10 days exhibit very irregular shapes. Note also that, since the set of historical data used is relatively small, we do not reject extreme values or bins with low occupation numbers (as done for instance by Daniel, Joseph & Bree, 2005, Dragulescu & Yakovenko, 2002, Prange, Silva & Yakovenko, 2004, Silva & Yakovenko, 2001).

NUMERICAL RESULTS

In this section we present and discuss the numerical results obtained using the calibration method described in Section 4.

The values of the parameters $\vartheta, \gamma, \kappa, \eta, \mu, v_0$ obtained for the six stocks considered are shown in Table 1. Note that in Table 1, as well as in the remainder of the paper, the parameters $\vartheta, \gamma, \kappa, \eta, \mu, v_0$ are expressed in 1/*year* units.

In Figure 1 we compare, for the Pirelli & C. SpA stock, the probability density function of daily returns obtained using the HM (solid line) and the empirical probability density function of returns (dotted line). Note that the probability density function of returns of the HM is computed using formula (28), where the values of $\vartheta, \gamma, \kappa, \eta, \mu, v_0$ are those reported in the third row of Table 1, and $\Delta t = 1\,day$. Figure 1 shows also, for the Pirelli & C. SpA stock, the probability density function of returns obtained using the LM (dashed line), where the drift and volatility parameters are estimated using relations (9).

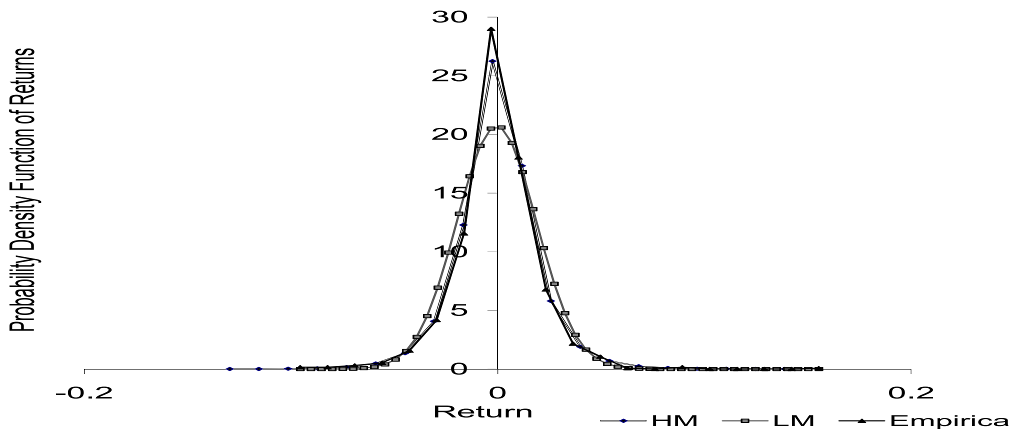
We may note that the HM provides an excellent agreement with historical data and fits the empirical probability distribution of returns far better than the LM. In particular the HM describes the high peaks of the empirical distribution significantly better than the LM. Moreover, although the historical dataset used is probably too small to obtain a sharp description of extreme events, the HM captures the fat tails of the empirical distribution better than the LM. Similar results are obtained also for the other stocks considered. As an example we show in Figure 2 the probability distributions of returns (HM, LM and empirical) obtained for the RaS Holding stock.

Table 1: Parameters of the HM

Number	Name	μ	v0	γ	θ	η	κ
1	Autostrade SpA	0.257	0.019	127.6	0.078	0.12	6.70
2	Mediobanca SpA	0.151	0.060	40.30	0.220	0.00	8.30
3	Pirelli & C. SpA	-0.058	0.080	27.5	0.320	0.00	8.90
4	RaS Holding	0.115	0.040	52.1	0.356	-0.07	11.3
5	Unicredito Italiano SpA	0.098	0.012	210.0	0.140	0.04	11.6
6	Snam Rete Gas	0.071	0.030	85.9	0.018	0.03	2.80

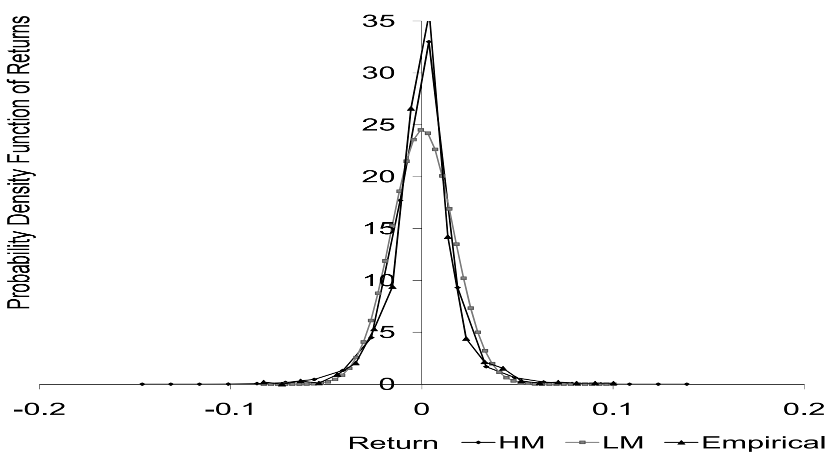
This table shows the values of the parameters of the HM obtained for six stocks traded on the Italian market.

Figure 1: Pirelli & C. SpA, Probability Distributions of Returns



This figure shows the probability density function of daily returns of the Pirelli & C. SpA stock.

Figure 2: RaS Holding, Probability Distributions of Returns



This figure shows the probability density function of daily returns of the RaS Holding stock.

Now let us consider the case of a portfolio composed by the six assets reported in Table 1. The six stocks considered are arranged in vector $\left[X_1(t), X_2(t), \ldots, X_6(t)\right]$, whose components appear in equation (22), following the same order as in Table 1. Using the algorithm described in Section 4, we estimate the following correlation matrix:

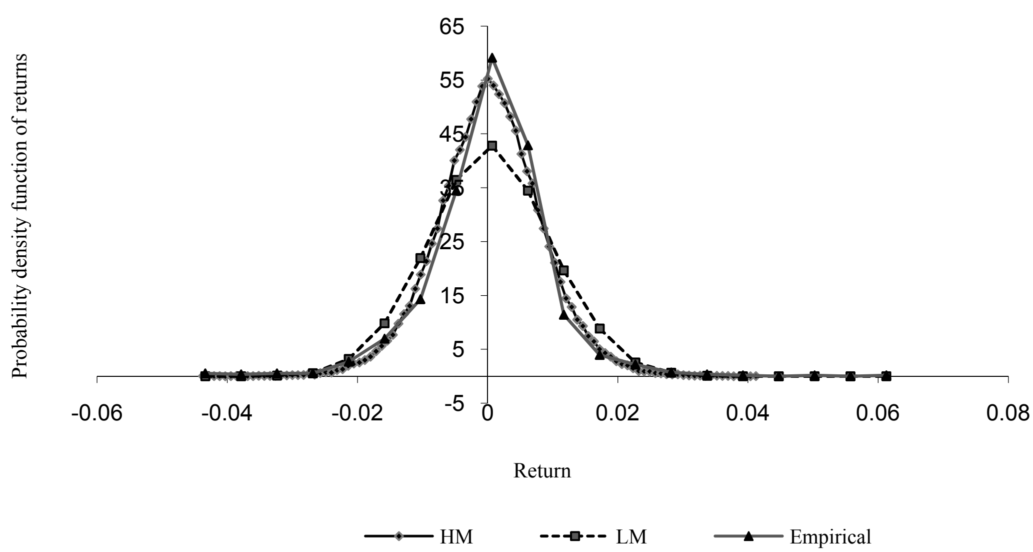
$$\Lambda = \begin{bmatrix}
1 & 0.1971 & 0.1088 & 0.1788 & 0.1590 & 0.1726 & 0.1200 & 0 & 0 & 0 & 0 & 0 \\
0.1971 & 1 & 0.2758 & 0.4793 & 0.5104 & 0.1137 & 0.0237 & 0 & 0 & 0 & 0 & 0 \\
0.1088 & 0.2758 & 1 & 0.2216 & 0.2504 & -0.0075 & 0.0131 & 0 & 0 & 0 & 0 & 0 \\
0.1788 & 0.4793 & 0.2216 & 1 & 0.5333 & 0.1627 & 0.0215 & 0 & 0 & -0.0700 & 0 & 0 \\
0.1590 & 0.5104 & 0.2504 & 0.5333 & 1 & 0.1082 & 0.0191 & 0 & 0 & -0.0254 & 0 & 0 \\
0.1726 & 0.1137 & -0.0075 & 0.1627 & 0.1082 & 1 & 0.0207 & 0 & 0 & -0.0092 & 0.0007 & 0.030 \\
0.1200 & 0.0237 & 0.0131 & 0.0215 & 0.0191 & 0.0207 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.0700 & -0.0254 & -0.0092 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0400 & 0.0007 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0300 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

The initial wealth of the portfolio has been equally distributed among the six stocks, so that every asset gives a relevant contribution to the value of the portfolio. The portfolio return at a given time t is measured as follows:

$$X_P = \log\left(\frac{P_t}{P_{t-1}}\right), \tag{36}$$

In Figure 3 we show the probability distribution of the portfolio returns obtained using the HM, the probability distribution of the portfolio returns obtained using the LM, and the probability distribution of the realized portfolio returns. Note that the probability distribution of both the HM and the LM is computed by Monte Carlo simulation. In particular in the Monte Carlo simulation of the HM, the stochastic differential equations (22), (23) are discretized in time using the Euler-Maruyama scheme (see Kloeden & Platen, 1999). We clearly note that the HM provides a very good description of the realized portfolio returns and fits the empirical distribution of the portfolio returns considerably better than the LM.

Figure 3: Portfolio of Six Stocks, Probability Distributions of Returns



This figure shows the probability density function of daily returns of a portfolio of six stocks traded on the Italian market.

CONCLUSIONS

We have investigated the performance of the Heston stochastic volatility model in describing the probability distribution of returns both in the case of single assets and in the case of asset portfolios. In particular we have proposed a simple method to calibrate the HM based on an integral representation of the exact probability density function of asset returns derived by Dragulescu & Yakovenko (2002). In the case of multiple correlated assets, the correlation parameters are estimated using an ad-hoc matrix completion algorithm.

Using the calibration method proposed in this paper the initial variance of asset returns is treated as an additional parameter of the model. Therefore it is not necessary to assume that the initial variance has stationary probability distribution (as done in previous works), and the value taken by the initial variance can be estimated.

We have used the calibration algorithm presented in this paper to estimate the parameters of the HM for several stocks traded on the Italian financial market. These numerical experiments reveal that, both in the case of single assets and in the case of asset portfolios, our calibration method provides an excellent agreement with historical time series data. Moreover the HM captures the kurtosis effect exhibited by the empirical probability distribution of returns and fits the empirical distribution of returns far better than the LM. Finally we remark that the calibration algorithm proposed in this paper is simple to implement, so we believe that it is very suitable for practical applications.

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