THEORETICAL AND NUMERICAL VALUATION OF CALLABLE BONDS

Dejun Xie, University of Delaware

ABSTRACT

This paper studies the value of a callable bond and the bond issuer's optimal financial decision regarding whether to continue the investment on the market or call the bond. Assume the market investment return follows a stochastic model, the value of contract is formulated as a partial differential equation system embedded with a free boundary, defining the level of market return rate at which it is optimal for the issuer to call the bond. A fundamental solution of the partial differential equation is derived, and used to formulate the value of the bond. A bisection scheme is implemented to solve the problem numerically.

JEL: D4, D46 G14, G15

KEYWORDS: Callable Bonds, optimal financial decision, stochastic model

INTRODUCTION

A callable bond is a bond that can be redeemed by the bond issuer prior to its maturity under certain conditions by paying off all the borrowing balance to the bond holders. Naturally the issuer shall invest the raised capital into market, expecting the return on the investment be higher than the amount he has to pay to the bond holder. A company will rationally choose to call a bond if it is paying a higher coupon while earning a lower return from market. Here we consider a callable bond with a duration $T > 0$ and a fixed coupon rate $c > 0$, where the issuer is allowed to call (redeem the bond) prematurely by paying off the borrowing at any time $d < T$. We assume the bond is fully amortized, i.e., the issuer pays an equal coupon payment $m > 0$ for each unit time period throughout the duration of the bond. This is in contrast to other types of bonds, say, zero-coupon bond, for instance, where the issuer does not pay back any amount of borrowing until the contract expires. The amortized callable bond is popular in practice mainly because payments are distributed into equal installments over the duration of the bond, which makes it possible for the issuer to make relatively smooth financial planning and avoid "payment shock". Denote by $P(d)$ the outstanding borrowing balance owed by the bond issuer to the bond holder at time d. If the coupon rate is zero, then $P(d)$ will linearly decrease in time from d=0 to $d = T$. However, this is unlikely the case in real economy since it would imply that the bond issuer uses for free the money borrowed from the bond buyers. The usual situation is c> 0, which means that bond issuer pays more, in sum of all the coupons, than its initial borrowing $P(0)$. For example, company A raised $1,000,000 as of today by issuing a 20-year callable bond with a fixed coupon rate c = 5%. Then the company needs to pay a coupon of m = 7.91 × 104 $/year to the bond holders. We would like to remark that in practice, a corporate bond sometimes only specifies m, $P(0)$, and T, leaving c as an embedded implicit term of the contract. For this reason, c is sometimes referred as the implied coupon rate or implied interest rate.

Now suppose the bond issuer has raised $P(0)$ dollars of capital at $d = 0$ and for each unit time period, he pays a coupon of m dollars back to the bond holder, and if he decides to call at time d, he must pay off the borrowing balance of P(d) dollars. Then the valuation of bond is of interests to both the bond issuer and the holder. On one hand, the holder or a third party investor may want to know the fair value of such a bond. For investment banks or security companies which hold large portfolio of such bonds, the value of these bonds may have significant impact on their credit ratings and asset performance. And in situations like company merge or liquidation, the valuation of these bonds often becomes necessary, as required by statutory accounting principles. On the other hand, from the bond issuer's point of view, he expects to wisely use the capital such that the investment return is higher, on average, than the bond coupons he has to pay. In reality, the issuer may have many ways to use the capital. Here for simplicity, we consider his overall return from all possible market investment, the return rate of which follows certain stochastic process. The optimal decision at any given time, from the bond issuer's point of view, depends on how much return can be earned if an equal amount of capital *P(d)* be invested in the market. Intuitively, the issuer should not choose to call unless the overall investment return rate is expected to be very low for certain amount of time. Hence, the issuer must, at every moment while the contract is in effect, monitor the market investment return and decide whether it is of his best interest to call immediately.

The paper is organized as follows. In Section 2 we provide a review of related literature. In Section 3 we introduce the model of the underlying rate of return for the option and the partial differential equation governing the value of the bond. In section 3, we formulate the integral equation representation of the value of the bond with a free boundary incorporated. In section 4, we apply the Fourier transform to derive the Green's function used in our integral formulation of the solution. In section 5, we implement a bisection scheme to solve the problem iteratively and present some specific numerical examples. In section 6, we present some concluding remarks on our approach and possible future directions.

LITERATURE REVIEW

A seminal work of bond valuation can be traced back to Merton (1974), where the equity of a firm is treated as an option on its assets, and the value of a pure discount bond of a firm is analyzed under the assumption that the asset value of the firm follows geometric Brownian motion. Because of the important role played by callable bonds in real economy, related problems have been studied by considerable literatures. Black and Cox (1976) extend the theoretical framework of Merton (1974) to the situation where the value of a firm "follows a diffusion process with instantaneous variance proportional to the square of the value". Geske (1977) provides a compound option approach for valuing debt of a firm where the debt is restricted to be a single issue of discrete coupon debt and the firm is assumed dividend free. With these assumptions, Geske shows that the value of the corporate debt can be computed as the difference between the total value of the firm and the value of the equity. These earlier works share a common limitation by assuming market interest rate as constant. In an attempt to address the interest rate effect on the value of bonds as well as other interest rate options, Hull and White (1990) propose to value these options using a class of market data consistent spot rate models. Specifically, they present two one-sate variable models and show that the differences between the European style bond prices computed for these two models are small under certain assumptions. In the sequential, Hull and White (1993) propose a general procedure to construct interest rate models for fitting a given set of initial market data. They also provide a computing framework involving the construction of a trinomial tree for the short rate, working back through which the value of bond can be obtained. Aiming to generalize the framework for term structure models, Heath, Jarrow and Morton (1992) directly impose a stochastic structure on the evolution of forward interest rate curve. The model, based on the theory of arbitrage asset pricing, is proposed for valuing the interest rate options, including corporate bonds, when lacking the information of market price of risk. One shortcoming of this model is that it poses more computational difficulties in implementation. In recent development, reduced form method is introduced to study the bond valuation problems where when default is allowed. According to Duffie and Singleton (1999), a defaultable bond can be valued by discounting its payment streams at a default adjusted interest rate through a risk neutral measure.

We would like to remark that the problems of pricing defautable or callable bonds, like many other option valuation problems, typically do not have closed form solutions, thus a great deal of efforts have been made to find numerical solutions as well as approximated analytical solutions. For instance, a numerical partial differential equation scheme is implemented by Brennan and Schwartz (1977), a binomial lattice model was applied by Li et al. (1995), a Monte Carlo simulation approach is proposed by Kind and Wilde (2003), and a finite difference method is recently tried by Breton and Ben-Ameur (2005). A quasi-closed form approximation formula is obtained by Tourruco et al. (2007) for a zero coupon bond under the Black-Karasinski model. One notices (see Buttler (1995), for instance) that usual numerical techniques such as finite difference or binomial method typically provide poor accuracy and stability, which are mainly caused by the complexity of the free boundary conditions. Monte Carlo simulation is easy to implement, but is not quite efficient for pricing callable bonds with its drawback of low convergence. More detailed critiques of these these methods can be found in Jiang (2005). To rectify such numerical shortcomings, an integral equation approach has been recently proposed by Chen and Chadam (2007) and Xie (2008) for American option pricing and related problems. The main idea is to formulate the value of the option under review as an appropriate integral form, on basis of which one can derive an efficient algorithm to solve the problem iteratively. The same idea is applied in this work. We first derive an integral representation of the bond value in terms of market return rate *x* at time *t,* then use it to design a bisection algorithm to iteratively solve for the optimal early call boundary. Then the bond value is recovered by numerical integral algorithm.

THE MODEL AND THE PROBLEM FORMULATION

In this work we assume that the return rate that the bond issuer can earn from market investment follows the Vasicek (1977) model, where the market return rate *r* is treated as a Markov process, governed by the stochastic differential equation

$$dr_t = k(\theta - r_t)dt + \sigma dW_t \qquad (1)$$

where W_t is the standard Brownian process. The Vasicek model is composed of one deterministic term and one random term. The deterministic term (also "the drift term") is chosen to produce the so called "mean-reverting" property. And the random term is to model the volatility caused by (possibly infinite) unpredictable factors. Using integrating factor method, as explained in Jiang (2005), for instance, one can solve the stochastic differential equation and get the explicit (stochastic) solution for the return rate at any time $t > d$.

$$r_t = \theta + e^{-k(t-d)}(r_d - \theta) + \sigma \int_d^t e^{-k(t-d-u)} dW_u$$

We are interested in the value of the callable bond at any given time and the corresponding return rate. For mathematical convenience, instead of using real time *d*, we introduce $t := T - d$. Financially t is the time to expiry of the bond contract (hereafter simply referred as the "time"). Let *V(x, t)* be the bond value at time *t* and the corresponding return rate *x*. Standard theory of mathematical tells us that *V(x, t)*, when the bond is not optimal to call, must satisfy the Black-Sholes type partial differential equation (hereafter referred as "PDE")

$$\frac{\partial V}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - k(\theta - x)\frac{\partial V}{\partial x} + xV = m. \qquad (2)$$

When it is optimal to call at certain time *t, V(x, t)* becomes *P(t)*. Let *h(t)* be the unknown optimal return level for the issuer to exercise the call, then we have

$$V(x, t) = P(t) \quad \text{for } x <= h(t), \tag{3}$$

and this automatically gives

$$V(x, 0) = 0. \tag{4}$$

It is also apparent that

$$V_x(x, t) = 0 \text{ for } x < h(t). \tag{5}$$

Since $V(x, t)$ is assumed at least twice differentiable in x, we know that

$$V_x(x,t) = 0 \qquad \text{for } x = h(t). \tag{6}$$

By a trivial free of arbitrage argument, one can show the

$$h(0) = c, \tag{7}$$

i.e., the optimal call boundary must starts at c as we look backward from the expiry date. Mathematically (2)-(7) constitute a free boundary problem we need to solve, where the free boundary $x = h(t)$, defining the optimal level of return rate at which the bond is to be called, separates the right half t-x plane into two regions. For the continuation region where $x > h(t)$, the bond contract is in effect and the value of the bond is governed by (2). For the early exercise region where $x < h(t)$, the bond is called and the bond holder gets back the borrowing balance of $P(t)$. Because it is the issuer, rather than the holder, has the choice to act in response to the market, the value of the contract is always less or equal than the loan balance. Thus the free boundary is where the value of the bond first reaches value of $P(t)$, i.e., it is optimal for the issuer to call the bond if and only if that value of the bond reaches $P(t)$ for the first time. To solve the free boundary problem (2)-(7), we first derive the fundamental solution of (2), then use it to formulate the solution $V(x, t)$.

MATHEMATICAL MOTIVATION

When handling a free boundary PDE system, it is often tempting to investigate if we can find the solution for the homogeneous PDE without boundary conditions. If this can be done, then the solution to the system can be formulated in terms of integral equations with boundary conditions incorporated. Without loss of generality, we assume $m = 1$. Define

$$L(V) := \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + k(\theta - x)\frac{\partial V}{\partial r} - xV, \tag{8}$$

We see that, according to (2.5) and (2.6) ,

$$\frac{\partial V}{\partial t} - L(V) = F(x,t), \qquad \forall x \in \mathbb{R},\ t > 0 \tag{9}$$

Where

$$F(x,t) = \begin{cases} 1, & for \quad x > h(t),\ t > 0 \\ \frac{x}{c} + (1 - \frac{x}{c})e^{-ct}, & for \quad x \le h(t),\ t > 0 \end{cases}$$

If we can find the solution, say *G(r,y,t,τ)* to the PDE defined in (8), then, by the Green's identity, we would be able to write the solution as

$$V(x,t) = \int_0^t \int_{-\infty}^{\infty} F(y,\tau) G(x,y,t,\tau) dyd\tau. \tag{10}$$

And then the following manipulations can be made to find the integral identities on which a numerical iteration scheme can be designed.

$$\begin{aligned} V &= \int_0^t \int_{-\infty}^{h(\tau)} (\frac{y}{c} + (1-\frac{y}{c}))e^{-c\tau} G(r,y,t,\tau)dyd\tau + \int_0^t \int_{h(\tau)}^{\infty} G(r,y,t,\tau)dyd\tau \\ &= \int_0^t \int_{-\infty}^{\infty} (\frac{y}{c} + (1-\frac{y}{c}))e^{-c\tau} G(r,y,t,\tau)dyd\tau \\ &\quad - \int_0^t \int_{h(\tau)}^{\infty} G(r,y,t,\tau)(1-\frac{y}{c})(e^{-c\tau}-1)dyd\tau \end{aligned}$$

Denote

$$I = \int_{-\infty}^{\infty} (\frac{y}{c} + (1-\frac{y}{c}))e^{-c\tau} G(x,y,t,\tau)dy$$

We are going to show that $I = P(t)$. To this end we do not evaluate I directly. Instead, we integrate the differential form of G with respect to y over the whole x-space. Let $s = t - \tau$. Because G, by the nature of being the fundamental solution to the PDE in (8), satisfies

$$G_s - \frac{\sigma^2}{2} G_{yy} + [k(\theta - y)G]_y + yG = 0 \tag{11}$$

and G decays exponentially in y, as will become apparent after we derive its explicit formula, we can apply integral by parts to integrate (11) with respect to y in the whole x-space, thus have

$$\frac{d}{ds}\int_R Gdy = -\int_R yGdy$$

Then

$$\begin{aligned} I &= e^{-c\tau}\int_R Gdy + \frac{1-e^{-c\tau}}{c}\int yGdy \\ &= e^{-c\tau}\int_R Gdy - \frac{1-e^{-c\tau}}{c}\frac{d}{ds}\int Gdy \\ &= -\frac{1}{c}\frac{d}{ds}\{(1-e^{-c\tau})\int_R Gdy\} \end{aligned}$$

Now, using the fact that $\int_R G dy = 1, \quad \tau = t$, we get

$$\int_0^t I d\tau = \int_0 -\frac{1}{c}\frac{d}{ds}\{(1-e^{-c\tau})\int_R G dy\} = \frac{1}{c}(1-e^{-ct}) \equiv \mathrm{P}(t).$$

Let

$$U(x,t) = \int_0^t \int_{h(\tau)}^{\infty} G(r,y,t,\tau)(1-\frac{y}{c})(e^{-c\tau}-1)dyd\tau, \tag{12}$$

we get

$$V(x,t) = P(t) - U(x, t) \tag{13}$$

And now it is straightforward to translate the boundary conditions (3-6) of $V(x, t)$ into boundary conditions of $U(x, t)$, i.e., when $x = h(t)$,

$$U(x, t) = 0, \tag{14}$$

$$U_x(x,t) = 0. \tag{15}$$

So far the analysis is carried out as if the solution were known. The following section is dedicated to finding the fundamental solution G using the Fourier Transform method.

DERIVATION OF THE FUNDAMENTAL SOLUTION

Let G be the fundamental solution associated with (8). For every (x, t) fixed,
Define the Fourier transform in the x variable by:

$$F[G(r,y,t,\tau)] = \int_{-\infty}^{\infty} G(r,y,t,\tau)e^{-i\lambda r}d\,r = \hat{G}(\lambda,y,t,\tau),$$

we have

$$\begin{cases} \dfrac{\partial \hat{G}}{\partial \tau} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 G}{\partial y^2} - k(\theta - y)\dfrac{\partial G}{\partial y} + (k-y)\hat{G} = 0, \quad \text{for} \quad \tau < t,\ y \in \Box \\ \hat{G}(\lambda, y, t, t-) = e^{-i\lambda y} \end{cases} \tag{16}$$

We postulate that admits a solution of the form

$$\hat{G}(r,y,t,t-) = e^{A(t,\tau,\lambda)+yB(t,\tau,\lambda)}. \tag{17}$$

with $A(t, T, \lambda)$ and $B(t,T, \lambda)$ to be determined shortly. Substituting (17) back into the partial differential equation in (16), we get

$$\begin{cases} A' + \frac{\sigma^2}{2}B^2 - k\theta B + k = 0 \\ B' + kB - 1 = 0 \\ A(t,t,\lambda) = 0 \\ B(t,t,\lambda) = -i\lambda \end{cases} \tag{18}$$

Using the method of separation of variables to solve the second differential equation in (18), we get

$$B = (-\frac{1}{k} - i\lambda)e^{k(t-\tau)} + \frac{1}{k} \tag{19}$$

Substitute (19) into the first differential equation in (18), we have

$$\begin{aligned} A &= -\int_\tau^t k\theta B ds + \int_\tau^t \frac{\sigma^2}{2}B^2 ds + \int_\tau^t k ds \\ &= -k\theta\int_\tau^t [(-\frac{1}{k} - i\lambda)e^{k(t-s)} + \frac{1}{k}]ds + \frac{\sigma^2}{2}\int_\tau^t [(-\frac{1}{k} - i\lambda)e^{k(t-s)} + \frac{1}{k}]^2 ds + k(t-\tau) \\ &= \theta\int_\tau^t [(1+i\lambda k)e^{k(t-s)} - 1]ds + \frac{\sigma^2}{2k^2}\int_\tau^t [(1+i\lambda k)e^{k(t-s)} - 1]^2 ds + k(t-\tau) \end{aligned}$$

Next, we want to collect terms for $A + yB$ according to the powers of λ. For this purpose, we write

$$\begin{aligned} A + yB &= \theta\int_\tau^t [(1+i\lambda k)e^{k(t-s)} - 1]ds \\ &\quad + \frac{\sigma^2}{2k^2}\int_\tau^t [(1+i\lambda k)e^{k(t-s)} - 1]^2 ds + k(t-\tau) - y[(\frac{1}{k} + i\lambda)e^{k(t-\tau)} - \frac{1}{k}] \\ &= -\alpha_2\lambda^2 + \alpha_1 i\lambda + \alpha_0\lambda \end{aligned}$$

A tedious but straightforward computation will lead to exact expressions of α_0, α_1, and α_2 as functions in *y, t* and τ. But the integral form expressions of α_0, α_1, and α_2 are good enough for computational purposes. Now we can apply the inverse of Fourier transform to derive the desired the solution

$$\begin{aligned} G(r,y,t,\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(r,y,t,\tau)e^{i\lambda r}d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha_2\lambda^2 + \alpha_1 i\lambda + \alpha_0\lambda}e^{i\lambda r}d\lambda \\ &= \frac{1}{\sqrt{4\pi\alpha_2}}e^{\alpha_0 - \frac{(\alpha_1 + r)^2}{4\alpha_2}} \end{aligned}$$

It is easy to verify that $G(x, y, t, \tau)$ is exponentially decaying in y for x, t, τ fixed, and the spatial integral with respect to y is always equal to 1 for fixed tau at t. Now the expression for $U_r(r, t)$ involves a double integral over an infinite domain, sometimes it is beneficial to reduce the double integral to a single one. This is because that the inside integral of U can be evaluated by, say, change of variables. Since the computation is tedious and such a further simplification is not necessary in finding numerical solutions, we omit these further computations.

NUMERICAL EXAMPLE AND DISCUSSION

Here we present some numerical examples obtained from our method. Our numerical algorithm is based on the free boundary condition defined in (14), where the integral representation of $U(x, t)$ is given by (12). Recall that $h(t)$ is defined as the level of return rate x at which $V(x, t)$ reaches $P(t)$ for the first time, or equivalently, $U(x, t)$ decreases to 0 for the first time. Start from $h(0) = c$ and $U(x, 0) = 0$, for each $t > 0$ fixed, we apply bisection scheme to find h(t), then use the integral representation (12) to recover $U(x, t)$, and use (13) to recover $V(x, t)$. To implement the bisection scheme, we start with the two initial guesses of $h(t)$. If they are too high compared to the true value of $h(t)$, we somehow shift downward the next guess; if too low, we somehow shift upward the next guess; if one is above and the other is below the true value, we take the average to update the guess. For a given bond with duration T and coupon rate c, assuming the model parameters are known, we partition the time [0, T] into N evenly spaced subintervals with $dt = T/N$. Define $t = (t_0, t_1, \ldots, t_N)$, a time vector with $t_n = ndt$, $n = 0, \ldots N$. For a prescribed error tolerance level, say, Tole=10^{-8}, we implement our algorithm as follows.

(1) $h(t_0) = c$, which is known from (7).

(2) Suppose we have found $h(t_1), h(t_2), \ldots, h(t_{n-1})$, we apply a bisection scheme to find $h(t_n)$:

(a) Take reasonable two initial guesses of $h(t_n)$, say $h^1(t_n)$ and $h^2(t_n)$. We assume $h^1(t_n) > h^2(t_n)$; if not, simply switch index 1 and 2.

(b) Apply numerical integration method, say Simpson quadrature, for instance, to evaluate the integral in (12), with $h(\tau)$ being interpolated by $(t_0, h(t_0))$, $(t_1, h(t_1))$, …, $(t_n, h^i(t_n))$, and get $U(h^i(t_n), t_n)$ accordingly for $i = 1, 2$, respectively.

(c) If $U(h^1(t_n), t_n) > 0$ and $U(h^2(t_n), t_n) > 0$, then we set

$$h^3(t_n) = h^2(t_n) - [h^1(t_n) - h^2(t_n)].$$

(d) If $U(h^1(t_n), t_n) < 0$ and $U(h^2(t_n), t_n) < 0$, then we set

$h^3(t_n) = h^1(t_n) + [h^1(t_n) - h^2(t_n)]$;

$h^2(t_n) = h^1(t_n)$.

(e) If $U(h^1(t_n), t_n) > 0$ and $U(h^2(t_n), t_n) < 0$, then we set

$h^3(t_n) = [h^1(t_n) + h^2(t_n)]/2$; 2

(i) If $U(h^3(t_n),\ t_n) > 0$, set

$$h^2(t_n) = h^2(t_n).$$

(ii) Otherwise, set

$$h^2(t_n) = h^1(t_n).$$

(f) If $|U(h^3(t_n),\ t_n)|$ <Tole, iteration ends. If not, use $h^2(t_n)$ and $h^3(t_n)$ as two updated initial guesses, repeat steps (a) through (f) to find $h^4\ (t_n)$. Repeat such an iteration until an index k is reached such that $|U(h^k(t_n),\ t_k)| <$ Tole.

(3) Once $h(t_n)$'s have been found for $n = 0,\ 1,\ \ldots,\ N$, $V(x,\ t_n)$ can be recovered by (13) for arbitrary given x using numerical integration quadratures.

For the iteration to converge faster, it is better to start with one initial guess above and the other below the true value of $h(t_n)$. Our numerical experiments show that the choices of c and $-c$ are good enough for most cases. Also to increase the accuracy of the numerical solution, one can increase N, the number of grids for partitioning the time interval [0, T]. For typical parameters with $T < 30$, our numerical simulations show that $N = 4096$ is large enough for achieving a solution with relative error less than 10^{-6}, where relative error is defined as the difference of numerical values of $h(T)$'s achieved with different N's.

As an example, consider a 5-year bond as of today with coupon payment $m = 1$ (dollars per year), and coupon rate $c = 0.08$. And the corresponding borrowing balance is $P(t) = 4.1210$ (dollars). Assume the parameter values appearing in (2.1) are $\theta = 0.04$, $k = 0.2$, $a = 0.01$, we implement our numerical method and get the bond value as of today V as a function of current market return rate x. The "$c = 0.08$" column in Table 1 is the output of V (in dollars) of such a bond for $x = 0.01, 0.02, ..., 0.12$. Indeed one can see that V increases as x decreases, when other variables and parameters fixed. To make optimal financial decision, the bond issuer needs to apply the algorithm to compute V and compare it with P. If the the current market return rate $x = 0.10$, say, then $V = 4.0419$ (dollars). Since $V(0.10, 5) = 4.0419 < 4.1210 = P(5)$, the issuer should not call the bond. Financially it means that cost of fund raising through issuance of bond is inexpensive enough that the issuer can benefit by investing $P(5)$ amount of capital to earn a relatively higher market return. On the other hand, if $x = 0.07$, then $V(0.07, 5) = P(5) = 4.1210$, and the issuer should call the bond. Financially it means that the cost of fund raising through issuance of bond is so expensive that it is a wise decision for the issuer to close the deal. And as time changes, the issuer shall obtain the updated return rate x, redo the computation, and make updated comparison. For similar bonds with coupon rate $c = 0.02, 0.04, 0.06$, one can similarly compute V for $x = 0.01, 0.02, ..., 0.12$. The results are presented in Table 1. Keeping other variables and parameters unchanged, one can run the program again for $t = 10$ and $t = 20$, for instance, and the outputs are tabulated in Table 2 and 3.

CONCLUSION AND DISCUSSION

Assuming the return rate of market investment follows the Vasicek model, we formulated and numerically solved a callable bond valuation problem. An exact solution of the governing PDE is obtained and used to derive the representation of the contract value. A bisection algorithm is implemented and validated to solve the problem numerically. Numerical simulations show that our algorithm is fast and stable.

Table 1: Numerical Computation of Bond Price *V(x,t)*

x	c = 0.02	c = 0.04	c = 0.06	c = 0.08
0.0100	4.7493	4.5317	4.3197	4.1210
0.0200	4.6890	4.5317	4.3197	4.1210
0.0300	4.6120	4.5317	4.3197	4.1210
0.0400	4.5321	4.5016	4.3197	4.1210
0.0500	4.4527	4.4405	4.3197	4.1210
0.0600	4.3746	4.3697	4.3083	4.1210
0.0700	4.2981	4.2962	4.2652	4.1210
0.0800	4.2232	4.2226	4.2068	4.1167
0.0900	4. 1500	4.1500	4. 1418	4.0877
0.1000	4.0783	4.0786	4.0744	4.0419
0.1100	4.0083	4.0087	4.0065	3.9871
0.1200	3.9398	3.9403	3.9390	3.9276

Table 1 presents a numerical computation of bond price V(x,t) at different market interest rate x and coupon rate c. Here the duration of bond is t=5, and the values for the model parameters are θ =0.04, k=0.2, σ = 0.01

Table 2: Numerical Computation of Bond Price *V(x,t), t =10*

x	c = 0.02	c = 0.04	c = 0.06	c = 0.08
0.0100	8.9075	8.2420	7.5198	6.8834
0.0200	8.6942	8.2420	7.5198	6.8834
0.0300	8.4704	8.2334	7.5198	6.8834
0.0400	8.2482	8.1254	7.5198	6.8834
0.0500	8.0310	7.9614	7.5198	6.8834
0.0600	7.8197	7.7778	7.4902	6.8834
0.0700	7.6147	7.5885	7.3944	6.8834
0.0800	7.4160	7.3991	7.2643	6.8747
0.0900	7.2235	7.2124	7.1167	6.8197
0.1000	7.0370	7.0297	6.9607	6.7312
0.1100	6.8564	6.8515	6.8011	6.6221
0.1200	6.6814	6.6782	6.6411	6.5003

Table 2 presents a numerical computation of bond price V(x,t) at different market interest rate x and coupon rate c. Here the duration of bond is t=10, and the values for the model parameters are θ =0.04, k=0.2, σ = 0.01

In addition to its mathematical robustness, the algorithm can be a useful tool for portfolio management purposes. Practitioners who wish to create maximum yield using borrowed capital should be able to apply our program to monitor the market condition and decide when it is optimal to liquidate its investment. While the method is designed for valuing the callable bond with market interest following the Vasicek model, we feel it can be extended to similar problems where market interest follows other mean-revering models. One limitation of our current work is the assumption that the issuer must settle the balance in whole amount if he decides to call the bond. As a future research direction, we would like to study the optimal prepayment strategy for the issuer if partial payments are allowed.

Table 3: Numerical Computation of Bond Price *V(x,t), t =20*

x	***c* = 0.02**	***c* = 0.04**	***c* = 0.06**	***c* = 0.08**
0.0100	15.3493	13.7668	11.6468	9.9763
0.0200	14.8234	13.7668	11.6468	9.9763
0.0300	14.3136	13.6967	11.6468	9.9763
0.0400	13.8221	13.4209	11.6468	9.9763
0.0500	13.3491	13.0641	11.6468	9.9763
0.0600	12.8943	12.6801	11.5902	9.9763
0.0700	12.4572	12.2900	11.4223	9.9763
0.0800	12.0370	11.9030	11.1944	9.9627
0.0900	11.6331	11.5236	10.9341	9.8784
0.1000	11.2450	11.1541	10.6568	9.7416
0.1100	10.8719	10.7957	10.3714	9.5712
0.1200	10.5133	10.4488	10.0837	9.3790

Table 3 presents a numerical computation of bond price V(x,t) at different market interest rate x and coupon rate c. Here the duration of bond is t=20, and the values for the model parameters are θ =0.04, k=0.2, σ = 0.01

REFERENCES

Buttler, H.J. (1995), "Evaluation of callable bonds: Finite difference methods, stability and accuracy", Economic Journal, vol. 105, 374-384.

Duffie, D. & Singleton, K.J. (1999), "Modeling term structures of defaultable bonds", Review of Financial Studies, vol. 12, 687-720.

Brennan, M. J. & Schwartz, E. S. (1977), "Convertible Bonds: Valuation and Optimal Strategies for Call and Conversion The Journal of Finance, vol. 32, 1699-1715

Chen X. & Chadam, J. (2007), "Mathematical analysis of an American put option", SIAM J. Math. Anal., vol. 38, 1613--1641

Xie, D. (2008), "A PDE approximation approach for the optimal early exercise boundary of American put option", Far East Journal of Applied Mathematics, vol. 33, 411-423

Hull, J. C. & White, A. (1990), "Pricing interest rate derivative securities", Review of Financial Studies, vol. 3, 573-592 .

Brennan, M. & Schwartz, E. (1977), "Savings bonds, retractable bonds, and callable bonds", Journal of Financial Economics, vol. 5, 67-88.

Vasicek, O. A. (1977), " An equilibrium characterization of the term structure", J. Fin. Econ, vol. 5, 177–188.

Kind, A. & Wilde, C. (2003), "A Simulation-Based Pricing Method for Convertible Bonds", Derivatives Research, New York University Faculty Digital Archive, S-DRP-03-07

Tourruco, F., Hagan , P. S. & Schleiniger, G. F. (2007), "Approximate Formulas for Zero-coupon Bonds", Applied Mathematical Finance, vol. 14, 207-226

Merton, R. (1974), "On the pricing of corporate debt: the risk of structure of interest rates", Journal of Finance, vol. 29, 449-469.

Breton, M. and Ben-Ameur H. (2005), "A Finite Element Method for Two Factor Convertible Bonds", Numerical Methods in Finance, Springer US, 109-128

Kau, J. B. & Keenan, D.C. (1995), "An Overview of the Option-Theoretic Pricing of Mortgages", J. of Housing Res., vol. 6, 217-244.

Jiang, L. (2005), "Mathematical Modeling and Methods of Option Pricing", World Scientific Publishing Company

Li, A., Ritchken, P & Subranmanian, S. (1995), "Lattice Models for Pricing American Interest Rate Claims", Journal of Finance, vol. 50, 719-737

Black, F. and Cox, J. C. (1976), "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions", Journal of Finance, vol. 31, 351-367

Geske, R. (1977), "The Valuation of Corporate Liabilities as Compound Options", Journal of Financial and Quantitative Economics, vol. 12, 541-552.

Hull, J. and White, A. (1993), "One Factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities", Journal of Financial and Quantitative Analysis, vol. 28, 235-254.

Heath D., Jarrow, R. and Morton, A. (1992), "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation", Econometrica, vol. 60, 77-105.

BIOGRAPHY

Dejun Xie is a UNIDEL Researcher in the Department of Mathematical Science at the University of Delaware. He holds a Ph.D. in Applied Mathematics from the University of Pittsburgh. His current research interests include quantitative economics and mathematical finance.