APPLICATION OF A HIGH-ORDER ASYMPTOTIC EXPANSION SCHEME TO LONG-TERM CURRENCY OPTIONS

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ABSTRACT

Recently academic researchers and practitioners have use the asymptotic expansion method to examine a variety of financial issues under high-dimensional stochastic environments. This methodology is mathematically justified by Watanabe theory (Watanabe, 1987), and Malliavin calculus (Yoshida, 1992a,b) and essentially based on the framework initiated by Kunitomo and Takahashi (2003) and Takahashi (1995, 1999) in a financial context. In practical applications, it is desirable to investigate the accuracy and stability of the method especially with expansion to higher orders in situations where the underlying processes are highly volatile. After Takahashi (1995,1999) and Takahashi and Takehara (2007) provided explicit formulas for the expansion to the third order, Takahashi, Takehara and Toda (2009) develop general computation schemes and formulas for an arbitrary-order expansion under general diffusion-type stochastic environments. In this paper, we describe these techniques in a simple setting to illustrate thier key ideas. To demonstrate their effectiveness the techniques are applied to pricing long-term currency options.

JEL: C63, G13

KEYWORDS: Asymptotic Expansion, Malliavin Calculus, Stochastic Volatility, Libor Market Model, Currency Options

INTRODUCTION

T his paper explains two alternative computation schemes proposed by Takahashi, Takehara and Toda (2009). The work is based on the asymptotic expansion approach based on Watanabe theory (Watanabe, 1987) in Malliavin calculus. The explanation is provided in a simple setting and applied to pricing long-term currency options under a cross-currency Libor market model and general stochastic volatility of spot exchange rates.

Recently, academic researchers and practitioners have used the asymptotic expansion method for a variety of financial issues. e.g. pricing or hedging complex derivatives under high-dimensional underlying stochastic environments. These methods are fully or partially based on the framework developed by Kunitomo and Takahashi (1992), Takahashi (1995,1999) in a financial literature. In theory, this method provides the expansion of underlying stochastic processes. This has a proper meaning in the limit of some ideal situations including deterministic cases (for details see Watanabe, 1987; Yoshida, 1992a; or Kunitomo and Takahashi, 2003).

In practice, however, researchers are often interested in cases far from the ideal, where the underlying processes are highly volatile as seen in recent financial markets. From the view point of accuracy and stability in practical uses, it is desirable to investigate behaviors of estimators with expansion to high orders.

In asymptotic expansion applications, the crucial step is computation of conditional expectations appearing in expansions, especially in expansion to high orders which is important in cases with long maturities or/and with highly volatile underlying variables. Takahashi, Takehara and Toda (2009) developed two alternative schemes for these computations in a general diffusion-type stochastic environment.

This paper describes the essence of their method in a much simpler setting and applies them to the evaluation of long-term currency options with maturities up to twenty years under a cross-currency Libor market model and general stochastic volatility of a spot exchange rates. It is very complex to obtain closed-form formulas in this instance. The remainder of the paper is as follows: In the following section we discuss the relevant literature. Next our methods are developed in simple setting, Section 3 applies the algorithms described in the previous section to concrete financial models, and confirms the effectiveness of the higher order expansions by numerical example. Detailed proofs, formulas and argument of the applied technique in a general setting including our complex example are found in Takahashi, Takehara and Toda (2009).

LITERATURE REVIEW

In this subsection we briefly review literature related to asymptotic expansion. The first known application of asymptotic expasion based on Watanabe theory in finance was Kunitomo and Takahashi (1992) who evaluated average options. Kunitomo and Takahashi (1992) and Takahashi (1995) derive approximation formulas for an average option by an asymptotic method. Their method is based on log-normal approximations of an average price distribution when the underlying asset price follows a geometric Brownian motion process. Yoshida (1992b) applies a formula derived by the asymptotic expansion of certain statistical estimators for small diffusion processes.

Thereafter asymptotic expansion has been applied to a broad class of problems in finance. In a general setting, the basic framework of the method was described in Kunitomo and Takahashi (2003), Takahashi (1999, 2009). Kunitomo and Takahashi (2001) generalized and applied the method to interest derivatives where the underlying model was not necessarily Markovian. Matsuoka, Takahshi and Uchida (2004) computed Greeks, the sensitivities of derivatives with respect to parameters. In Takahashi and Yoshida (2004, 2005) the method was used for the optimal portfolio problem and a new variance reduction technique for Monte Carlo simulations with the asymptotic expansion was developed. Muroi (2005) considered credit derivatives. Pricing currency options under the cross-currency Libor market model and exchange rates with sotchastic volatility and/or jumps, were examined in Takahashi and Takehara (2007, 2008a,b). Takahashi, Takehara and Toda [2009] introduced genral procedures for actual computation in the method which are applied in this paper.

AN ASYMPTOTIC EXPANSION APPROACH IN A BLACK-SCHOLES ECONOMY

In this section, we explain the concepts of this paper in a simple Black-Scholes-type economy. Let (W, P) be a one-dimensional Wiener space. Hereafter P is considered as a risk-neutral equivalent martingale measure and a risk-free interest rate is set to be zero for simplicity. Then, the underlying economy is specified with a ($\mathbf{R}_+$-valued) single risky asset $S^{(\varepsilon)} = \{S_t^{(\varepsilon)}\}$ satisfying:

$$S_t^{(\varepsilon)} = S_0 + \varepsilon \int_0^t \sigma(S_s^{(\varepsilon)}, s) dW_s \tag{1}$$

where $\varepsilon \in (0,1]$ is a constant parameter; $\sigma : \mathbf{R}_+^2 \mapsto \mathbf{R}$ satisfies some regularity conditions. We will consider the following pricing problem;

$$V(0,T) = \mathbf{E}[\Phi(S_T^{(\varepsilon)})] \tag{2}$$

where Φ is a payoff function written on $S_T^{(\varepsilon)}$ (for example, $\Phi(x) = \max(x - K, 0)$ for call options or $\Phi(x) = \delta_x(x)$, a delta function with mass at x for the density function) and $\mathbf{E}[\cdot]$ is an expectation operator under the probability measure P. Rigorously speaking, they are a generalized function of the Wiener function $S^{(\varepsilon)}$ and a generalized expectation for generalized functions, whose mathematically proper definitions are given in Section 2 of Takahashi, Takehara and Toda (2009).

Let $A_{kt} = \frac{\partial^k S_t^{(\varepsilon)}}{\partial \varepsilon^k}\big|_{\varepsilon=0}$. Here we represent A_{1t}, A_{2t} and A_{3t} explicitly by

$$A_{1t} = \int_0^t \sigma(S_s^{(0)}, s) dW_s, \tag{3}$$

$$A_{2t} = 2\int_0^t \partial\sigma(S_s^{(0)}, s) A_{1s} dW_s, \tag{4}$$

$$A_{3t} = 3\int_0^t \left(\partial^2\sigma(S_s^{(0)}, s)(A_{1s})^2 + \partial\sigma(S_s^{(0)}, s)(A_{2s}) \right) dW_s \tag{5}$$

recursively and then $S_T^{(\varepsilon)}$ has its asymptotic expansion

$$S_T^{(\varepsilon)} = S_0 + \varepsilon A_{1T} + \frac{\varepsilon^2}{2!} A_{2T} + \frac{\varepsilon^3}{3!} A_{3T} + o(\varepsilon^3). \tag{6}$$

Note that $S_t^{(0)} = \lim_{\varepsilon \downarrow 0} S_t^{(\varepsilon)} = S_0$ for all t. Next, normalize $S_T^{(\varepsilon)}$ with respect to ε as $G^{(\varepsilon)} = \frac{S_T^{(\varepsilon)} - S_T^{(0)}}{\varepsilon}$ for $\varepsilon \in (0,1]$. Then,

$$G^{(\varepsilon)} = A_{1T} + \frac{\varepsilon}{2!} A_{2T} + \frac{\varepsilon^2}{3!} A_{3T} + o(\varepsilon^2) \quad \text{in } L^P \text{ for every } p > 1.$$

Here the following assumption is made: $\Sigma_T = \int_0^T \sigma^2(S_t^{(0)}, t) dt > 0$. Note that A_{1T} follows a normal distribution with mean 0 and variance Σ_T, implying that the distribution of A_{1T} does not degenerate. It is clear that this assumption is satisfied when $\sigma(S_t^{(0)}, t) > 0$ for some $t > 0$. Then, the expectation of $\Phi(G^{(\varepsilon)})$ is expanded around $\varepsilon = 0$ up to ε^2-order in the sense of Watanabe (1987) and Yoshida (1992a) as follows. Hereafter the asymptotic expansion of $\mathbf{E}[\Phi(G^{(\varepsilon)})]$ up to the second order will be considered:

$$\mathbf{E}[\Phi(G^{(\varepsilon)})] = \mathbf{E}\left[\Phi(A_{1T})\right] + \varepsilon\mathbf{E}\left[\Phi^{(1)}(A_{1T})A_{2T}\right] + \varepsilon^2\left\{\mathbf{E}\left[\Phi^{(1)}(A_{1T})A_{3T}\right] + \frac{1}{2}\mathbf{E}\left[\Phi^{(2)}(A_{1T})(A_{2T})^2\right]\right\} + o(\varepsilon^2)$$

$$= \mathbf{E}\left[\Phi(A_{1T})\right] + \varepsilon\mathbf{E}\left[\Phi^{(1)}(A_{1T})\mathbf{E}\left[A_{2T} \mid A_{1T}\right]\right]$$

$$+ \varepsilon^2\left\{\mathbf{E}\left[\Phi^{(1)}(A_{1T})\mathbf{E}\left[A_{3T} \mid A_{1T}\right]\right] + \frac{1}{2}\mathbf{E}\left[\Phi^{(2)}(A_{1T})\mathbf{E}\left[(A_{2T})^2 \mid A_{1T}\right]\right]\right\} + o(\varepsilon^2)$$

$$= \int_{\mathbf{R}} \Phi(x) f_{A_{1T}}(x) dx + \varepsilon\int_{\mathbf{R}} \Phi^{(1)}(x)\mathbf{E}\left[A_{2T} \mid A_{1T} = x\right] f_{A_{1T}}(x) dx$$

$$+ \varepsilon^2\left\{\int_{\mathbf{R}} \Phi^{(1)}(x)\mathbf{E}\left[A_{3T} \mid A_{1T} = x\right] f_{A_{1T}}(x) dx + \frac{1}{2}\int_{\mathbf{R}} \Phi^{(2)}(x)\mathbf{E}\left[(A_{2T})^2 \mid A_{1T} = x\right] f_{A_{1T}}(x) dx\right\} + o(\varepsilon^2)$$

$$= \int_{\mathbf{R}} \Phi(x) f_{A_{1T}}(x) dx + \varepsilon \int_{\mathbf{R}} \Phi(x)(-1) \frac{\partial}{\partial x} \{ \mathbf{E}\left[A_{2T} \middle| A_{1T} = x \right] f_{A_{1T}}(x) \} dx$$

$$+ \varepsilon^2 \left(\int_{\mathbf{R}} \Phi(x)(-1) \frac{\partial}{\partial x} \{ \mathbf{E}\left[A_{3T} \middle| A_{1T} = x \right] f_{A_{1T}}(x) \} dx \right.$$

$$\left. + \frac{1}{2} \int_{\mathbf{R}} \Phi(x)(-1)^2 \frac{\partial^2}{\partial x^2} \{ \mathbf{E}\left[(A_{2T})^2 \middle| A_{1T} = x \right] f_{A_{1T}}(x) \} dx \right) + o(\varepsilon^2). \tag{8}$$

where $\Phi^{(m)}(x)$ is m-th order derivative of $\Phi(x)$ and $f_{A_{1T}}(x)$ is a probability density function of A_{1T} following a normal distribution; $f_{A_{1T}}(x) := \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{x^2}{2\Sigma_T}\right)$. In particular, letting $\Phi = \delta_x$, we have the asymptotic expansion of the density function of $G^{(\varepsilon)}$ as seen later. Then, to evaluate this expansion a computation of these conditional expectations is completed. Specifically, we present two alternative approaches.

An Approach with an Expansion into Iterated Itô Integrals

In this subsection we show an approach with further expansion of A_{2T}, A_{3T} and $(A_{2T})^2$ into iterated Itô integrals to compute the conditional expectations in (8). Recall that we have:

$$\mathbf{E}[\Phi(G^{(\varepsilon)})] = \int_{\mathbf{R}} \Phi(x) f_{A_{1T}}(x) dx + \varepsilon \int_{\mathbf{R}} \Phi(x)(-1) \frac{\partial}{\partial x} \{ \mathbf{E}\left[A_{2T} \middle| A_{1T} = x \right] f_{A_{1T}}(x) \} dx$$

$$+ \varepsilon^2 \left(\int_{\mathbf{R}} \Phi(x)(-1) \frac{\partial}{\partial x} \{ \mathbf{E}\left[A_{3T} \middle| A_{1T} = x \right] f_{A_{1T}}(x) \} dx \right.$$

$$\left. + \frac{1}{2} \int_{\mathbf{R}} \Phi(x)(-1)^2 \frac{\partial^2}{\partial x^2} \{ \mathbf{E}\left[(A_{2T})^2 \middle| A_{1T} = x \right] f_{A_{1T}}(x) \} dx \right) + o(\varepsilon^2).$$

$$\tag{9}$$

Next, it is shown that A_{2T}, A_{3T}, $(A_{2T})^2$ can be expressed as summations of iterated Itô integrals. First, note that A_{2T} is:

$$A_{2T} = 2 \int_0^T \int_0^{t_1} \partial\sigma(S_{t_1}^{(0)}, t_1) \sigma(S_{t_2}^{(0)}, t_2) dW_{t_2} dW_{t_1} \tag{10}$$

Next, by application of Itô's formula to (5) we obtain

$$A_{3T} = 6 \int_0^T \int_0^{t_1} \int_0^{t_2} \partial\sigma(S_{t_1}^{(0)}, t_1) \partial\sigma(S_{t_2}^{(0)}, t_2) \sigma(S_{t_3}^{(0)}, t_3) dW_{t_3} dW_{t_2} dW_{t_1}$$

$$+ 6 \int_0^T \int_0^{t_1} \int_0^{t_2} \partial^2\sigma(S_{t_1}^{(0)}, t_1) \sigma(S_{t_2}^{(0)}, t_2) \sigma(S_{t_3}^{(0)}, t_3) dW_{t_3} dW_{t_2} dW_{t_1} + 3 \int_0^T \int_0^{t_1} \partial^2\sigma(S_{t_1}^{(0)}, t_1) \sigma^2(S_{t_2}^{(0)}, t_2) dt_2 dW_{t_1}.$$

$$\tag{11}$$

Similarly,

$$(A_{2T})^2 = 16 \int_0^T \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \partial\sigma(S_{t_1}^{(0)}, t_1) \partial\sigma(S_{t_2}^{(0)}, t_2) \sigma(S_{t_3}^{(0)}, t_3) \sigma(S_{t_4}^{(0)}, t_4) dW_{t_4} dW_{t_3} dW_{t_2} dW_{t_1}$$

$$+ 8 \int_0^T \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \partial\sigma(S_{t_1}^{(0)}, t_1) \sigma(S_{t_2}^{(0)}, t_2) \partial\sigma(S_{t_3}^{(0)}, t_3) \sigma(S_{t_4}^{(0)}, t_4) dW_{t_4} dW_{t_3} dW_{t_2} dW_{t_1}$$

$$+8\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S_{t_1}^{(0)},t_1)\partial\sigma(S_{t_2}^{(0)},t_2)\sigma^2(S_{t_3}^{(0)},t_3)dt_3dW_{t_2}dW_{t_1}$$

$$+8\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S_{t_1}^{(0)},t_1)\partial\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_3}^{(0)},t_3)dW_{t_3}dt_2dW_{t_1}$$

$$+8\int_0^T\int_0^{t_1}\int_0^{t_2}\left(\partial\sigma(S_{t_1}^{(0)},t_1)\right)^2\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_3}^{(0)},t_3)dW_{t_3}dW_{t_2}dt_1\ +4\int_0^T\int_0^{t_1}\left(\partial\sigma(S_{t_1}^{(0)},t_1)\right)^2\sigma^2(S_{t_2}^{(0)},t_2)dt_2dt_1.$$

$$(12)$$

Then, by Proposition 1 in Takahashi, Takehara and Toda (2009), the conditional expectations in (9) can be computed as

$$\mathbf{E}[A_{2T}\mid A_{1T}=x]=2F_2\left(\partial\sigma^{(0)}\sigma^{(0)},(\sigma^{(0)})^2\right)\frac{H_2(x;\Sigma_T)}{\Sigma_T^2}=:c_2^{2,1}H_2(x;\Sigma_T) \qquad (13)$$

$$\mathbf{E}[A_{3T}\mid A_{1T}=x]=\left(6F_3\left(\partial\sigma^{(0)}\sigma^{(0)},\partial\sigma^{(0)}\sigma^{(0)},(\sigma^{(0)})^2\right)+6F_3\left(\partial^2\sigma^{(0)}\sigma^{(0)},(\sigma^{(0)})^2,(\sigma^{(0)})^2\right)\right)\frac{H_3(x;\Sigma_T)}{\Sigma_T^3}$$

$$+3F_2\left(\partial\sigma^{(0)}\sigma^{(0)},(\sigma^{(0)})^2\right)\frac{H_1(x;\Sigma_T)}{\Sigma_T} \qquad (14)$$

$$=:c_3^{3,1}H_3(x;\Sigma_T)+c_1^{3,1}H_1(x;\Sigma_T)$$

and

$$\mathbf{E}[(A_{2T})^2\mid A_{1T}=x]$$

$$=\left(16F_4\left(\partial\sigma^{(0)}\sigma^{(0)},\partial\sigma^{(0)}\sigma^{(0)},(\sigma^{(0)})^2,(\sigma^{(0)})^2\right)+8F_4\left(\partial\sigma^{(0)}\sigma^{(0)},(\sigma^{(0)})^2,\partial\sigma^{(0)}\sigma^{(0)},(\sigma^{(0)})^2\right)\right)\frac{H_4(x;\Sigma_T)}{\Sigma_T^4}$$

$$+\left(16F_3\left(\partial\sigma^{(0)}\sigma^{(0)},\partial\sigma^{(0)}\sigma^{(0)},(\sigma^{(0)})^2\right)+8F_3\left((\partial\sigma^{(0)})^2,(\sigma^{(0)})^2,(\sigma^{(0)})^2\right)\right)\frac{H_2(x;\Sigma_T)}{\Sigma_T^2}$$

$$+4F_3\left((\partial\sigma^{(0)})^2,(\sigma^{(0)})^2\right)H_0(x;\Sigma_T)\ =:c_4^{2,2}H_4(x;\Sigma_T)+c_2^{2,2}H_2(x;\Sigma_T)+c_0^{2,2}H_0(x;\Sigma_T) \qquad (15)$$

where $H_n(x;\Sigma)$ is a n-th order Hermite polynomial defined by

$$H_n(x;\Sigma):=(-\Sigma)^n e^{x^2/2\Sigma}\frac{d^n}{dx^n}e^{-x^2/2\Sigma},$$

with notations $F_n(f_1,\cdots,f_n):=\int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}}f_1(t_1)\cdots f_n(t_n)dt_n\cdots dt_1, n\geq 1,$

$\sigma^{(0)}=\sigma(S_t^{(0)},t)$ and $\partial^i\sigma^{(0)}=\partial^i\sigma(S_t^{(0)},t)$.

Substituting these into (9), we have the asymptotic expansion of $\mathbf{E}\left[\Phi(G^{(\varepsilon)})\right]$ up to ε^2-order. Further, letting $\Phi=\delta_x$, we have the expansion of $f_{G^{(\varepsilon)}}$, the density function of $G^{(\varepsilon)}$:

$$f_{G^{(\varepsilon)}}=f_{A_{1T}}(x)+\varepsilon(-1)\frac{\partial}{\partial x}\{\mathbf{E}\left[A_{2T}\mid A_{1T}=x\right]f_{A_{1T}}(x)\}$$

$$+\varepsilon^2\left((-1)\frac{\partial}{\partial x}\{\mathbf{E}\left[A_{3T}\mid A_{1T}=x\right]f_{A_{1T}}(x)\}+\frac{1}{2}(-1)^2\frac{\partial^2}{\partial x^2}\{\mathbf{E}\left[(A_{2T})^2\mid A_{1T}=x\right]f_{A_{1T}}(x)\}\right)+o(\varepsilon^2)$$

$$= f_{A_{1T}}(x) + \varepsilon(-1)\frac{\partial}{\partial x}\{c_2^{2,1}H_2(x;\Sigma_T)f_{A_{1T}}(x)\} \tag{16}$$

$$+\varepsilon^2\left((-1)\frac{\partial}{\partial x}\{\sum_{i=1,3}c_i^{3,1}H_i(x;\Sigma_T)f_{A_{1T}}(x)\} + \frac{1}{2}(-1)^2\frac{\partial^2}{\partial x^2}\{\sum_{i=0,2,4}c_i^{2,2}H_i(x;\Sigma_T)f_{A_{1T}}(x)\}\right) + o(\varepsilon^2).$$

An Alternative Approach with a System of Ordinary Differential Equations

In this subsection, we present an alternative approach in which the conditional expectations are computed through some system of ordinary differential equations. Again the asymptotic expansion of $\mathbf{E}\left[\Phi(G^{(\varepsilon)})\right]$ up to ε^2-order is considered. Note that the expectations of A_{2T}, A_{3T} and $(A_{2T})^2$ conditional on A_{1T} are expressed by linear combinations of a finite number of Hermite polynomials as in (13), (14) and (15). Thus, by Lemma 4 in Takahashi, Takehara and Toda (2009), we have we have

$$\mathbf{E}[A_{2T} \mid A_{1T} = x] = \sum_{n=0}^{2} a_n^{2,1} H_n(x;\Sigma_T), \tag{17}$$

$$\mathbf{E}[A_{3T} \mid A_{1T} = x] = \sum_{n=0}^{3} a_n^{3,1} H_n(x;\Sigma_T), \tag{18}$$

and $\mathbf{E}[(A_{2T})^2 \mid A_{1T} = x] = \sum_{n=0}^{4} a_n^{2,2} H_n(x;\Sigma_T), \tag{19}$

where the coefficients are given by

$$a_n^{2,1} = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial \xi^n}\bigg|_{\xi=0}\{\mathbf{E}[Z_T^{<\xi>}A_{2T}]\}, a_n^{3,1} = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial \xi^n}\bigg|_{\xi=0}\{\mathbf{E}[Z_T^{<\xi>}A_{3T}]\},$$

$$a_n^{2,2} = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial \xi^n}\bigg|_{\xi=0}\{\mathbf{E}[Z_T^{<\xi>}(A_{2T})^2]\}, \text{and} Z_t^{<\xi>} := \exp\left(i\xi A_{1t} + \frac{\xi^2}{2}\Sigma_t\right).$$

Note that $Z^{<\xi>}$ is a martingale with $Z_0^{<\xi>} = 1$. Since these conditional expectations can be represented by linear combinations of Hermite polynomials as seen in the previous subsection, the following should hold, which can be confirmed easily with results of this subsection:

$$\begin{cases} a_2^{2,1} = c_2^{2,1}; a_1^{2,1} = a_0^{2,1} = 0; a_3^{3,1} = c_3^{3,1}; a_1^{3,1} = c_1^{3,1}; a_2^{3,1} = a_0^{2,1} = 0; (20) \\ \quad a_4^{2,2} = c_4^{2,2}; a_2^{2,2} = c_2^{2,2}; a_0^{2,2} = c_0^{2,2}; a_3^{2,2} = a_1^{2,2} = 0. \end{cases} \tag{21}$$

Then, computation of these conditional expectations is equivalent to that of the unconditional expectations $\mathbf{E}[Z_T^{<\xi>}A_{2T}]$, $\mathbf{E}[Z_T^{<\xi>}A_{3T}]$ and $\mathbf{E}[Z_T^{<\xi>}(A_{2T})^2]$. First, applying Itô's formula to $\left(Z_t^{<\xi>}A_{2t}\right)$ we have

$$\mathbf{E}\left[Z_t^{<\xi>}A_{2t}\right] = \mathbf{E}\left[\int_0^t Z_s^{<\xi>}dA_{2s} + \int_0^t A_{2s}dZ_s^{<\xi>} + \left\langle A_2, Z^{<\xi>}\right\rangle_t\right]$$

$$= 2(i\xi)\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{1s}\right]ds \tag{22}$$

Then, applying Itô's formula to $\left(Z_t^{<\xi>}A_{1t}\right)$ again, we also have

$$\mathbf{E}\left[Z_t^{<\xi>}A_{1t}\right] = \mathbf{E}\left[\int_0^t Z_s^{<\xi>}dA_{1s} + \int_0^t A_{1s}dZ_s^{<\xi>} + \left\langle A_1, Z^{<\xi>}\right\rangle_t\right]$$

$$= (i\xi)\int_0^t \sigma^2(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}\right]ds = (i\xi)\int_0^t \sigma^2(S_s^{(0)},s)ds \qquad (23)$$

since $\mathbf{E}\left[Z_t^{<\xi>}\right]=1$ for all t.

Similarly, the following are obtained:

$$\mathbf{E}\left[Z_t^{<\xi>}A_{3t}\right] = 3(i\xi)\left(\int_0^t \partial^2\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}(A_{1s})^2\right]ds\right.$$

$$\left.+\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{2s}\right]ds\right) \qquad (24)$$

$$\mathbf{E}\left[Z_t^{<\xi>}(A_{1t})^2\right] = \int_0^t \sigma^2(S_s^{(0)},s)ds + 2(i\xi)\int_0^t \sigma^2(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{1s}\right]ds \qquad (25)$$

$$\mathbf{E}\left[Z_t^{<\xi>}(A_{2t})^2\right] = 4\int_0^t \left(\partial\sigma(S_s^{(0)},s)\right)^2\mathbf{E}\left[Z_s^{<\xi>}(A_{1s})^2\right]ds + 4(i\xi)\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{2s}A_{1s}\right]ds \qquad (26)$$

$$\mathbf{E}\left[Z_t^{<\xi>}A_{2t}A_{1t}\right] = 2\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{1s}\right]ds \qquad (27)$$

$$+(i\xi)\int_0^t (\sigma(S_s^{(0)},s))^2\mathbf{E}\left[Z_s^{<\xi>}A_{2s}\right]ds + 2(i\xi)\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}(A_{1s})^2\right]ds.$$

Then, $\mathbf{E}[Z_T^{<\xi>}A_{2T}]$, $\mathbf{E}[Z_T^{<\xi>}A_{3T}]$ and $\mathbf{E}[Z_T^{<\xi>}(A_{2T})^2]$ can be obtained as solutions of the system of ordinary differential equations (22), (23), (24), (25), (26) and (27). In fact, since they have a grading structure that the higher-order equations depend only on the lower ones, they can be easily solved by substituting each solution into the next ordinary differential equation recursively. Moreover, since these solutions are clearly the polynomial of $(i\xi)$, we can easily implement differentiations with respect to ξ in (17), (18) and (19). It is obvious that the resulting coefficients given by these solutions are equivalent to the results in the previous subsection.

In summary, in a Black-Scholes-type economy, we consider the risky asset $S^{(\varepsilon)}$ and evaluate some quantities, expressed as an expectation of the function of the terminal price, such as prices or risk sensitivities of the securities on the asset. First we expand them around the limit to $\varepsilon = 0$ so that we obtain the expansion (8) which contains some conditional expectations. Then, by approaches described in Section 2 and 3, we compute these conditional expectation. Finally, substituting computation results into (8), we obtain the asymptotic expansion of those quantities.

NUMERICAL EXAMPLES: APPLICATION TO LONG-TERM CURRENCY OPTIONS

In this section we apply our methods to pricing options on currencies under Libor Market Models(LMMs) of interest rates and stochastic volatility of the spot foreign exchange rate (Forex), which is much more complex then Black-Scholes-type case in the previous section. Due to limitation of space, only the structure of the stochastic differential equations of our model is described here. For details of the underlying model, see Takahashi and Takehara (2007). Detailed discussions in a general setting including the following examples are found in Section 3 and 4 of Takahashi, Takehara and Toda (2009).

Cross-Currency Libor Market Models

Let $(\Omega, F, \tilde{P}, \{F_t\}_{0 \le t \le T^* < \infty})$ be a complete probability space with a filtration satisfying the usual conditions. We consider the following pricing problem for the call option with maturity $T \in (0, T^*]$ and strike rate $K > 0$;

$$V^C(0; T, K) = P_d(0, T) \times \mathbf{E}^P\left[(S(T) - K)^+\right] = P_d(0, T) \times \mathbf{E}^P\left[(F_T(T) - K)^+\right] \tag{28}$$

where $V^C(0; T, K)$ denotes the value of a European call option at time 0 with maturity T and strike rate K, $S(T)$ denotes the spot exchange rate at time $t \ge 0$ and $F_T(t)$ denotes the time t value of the forex forward rate with maturity T. Similarly, for the put option we consider

$$V^P(0; T, K) = P_d(0, T) \times \mathbf{E}^P\left[(K - S(T))^+\right] = P_d(0, T) \times \mathbf{E}^P\left[(K - F_T(T))^+\right]. \tag{29}$$

It is well known that the arbitrage-free relation between the forex spot rate and the forex forward rate are given by $F_T(t) = S(t)\frac{P_f(t,T)}{P_d(t,T)}$ where $P_d(t, T)$ and $P_f(t, T)$ denote the time t values of domestic and foreign zero coupon bonds with maturity T respectively. $\mathbf{E}^P[\cdot]$ denotes an expectation operator under EMM(Equivalent Martingale Measure) P whose associated numeraire is the domestic zero coupon bond maturing at T.

For these pricing problems, a market model and stochastic volatility model are applied to modeling interest rates' and the spot exchange rate dynamics respectively. We first define domestic and foreign forward interest rates as $f_{dj}(t) = \left(\frac{P_d(t,T_j)}{P_d(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ and $f_{fj}(t) = \left(\frac{P_f(t,T_j)}{P_f(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ respectively, where $j = n(t), n(t)+1, \cdots, N$, $\tau_j = T_{j+1} - T_j$, and $P_d(t, T_j)$ and $P_f(t, T_j)$ denote the prices of domestic/foreign zero coupon bonds with maturity T_j at time t ($\le T_j$) respectively; $n(t) = \min\{i : t \le T_i\}$. We also define spot interest rates to the nearest fixing date denoted by $f_{d,n(t)-1}(t)$ and $f_{f,n(t)-1}(t)$ as $f_{d,n(t)-1}(t) = \left(\frac{1}{P_d(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)} - t)}$ and $f_{f,n(t)-1}(t) = \left(\frac{1}{P_f(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)} - t)}$. Finally, we set $T = T_{N+1}$ and will abbreviate $F_{T_{N+1}}(t)$ to $F_{N+1}(t)$ from here forward.

Under the framework of asymptotic expansion in the standard cross-currency libor market model, we consider the following system of stochastic differential equations(SDEs) under the domestic terminal measure P to price options. For detailed arguments on the framework of these SDEs see Takahashi and Takehara (2007).

As for the domestic and foreign interest rates we assume forward market models; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$f_{dj}^{(\varepsilon)}(t) = f_{dj}(0) + \varepsilon^2 \sum_{i=j+1}^{N} \int_0^t g_{di}^{0,(\varepsilon)}(u)' \gamma_{dj}(u) f_{dj}^{(\varepsilon)}(u) du + \varepsilon \int_0^t f_{dj}^{(\varepsilon)}(u) \gamma_{dj}'(u) dW_u, \tag{30}$$

$$f_{fj}^{(\varepsilon)}(t) = f_{fj}(0) - \varepsilon^2 \sum_{i=0}^{j} \int_0^t g_{fi}^{0,(\varepsilon)}(u)^{'} \gamma_{fj}(u) f_{fj}^{(\varepsilon)}(u) du + \varepsilon^2 \sum_{i=0}^{N} \int_0^t g_{di}^{0,(\varepsilon)}(u)^{'} \gamma_{fj}(u) f_{fj}^{(\varepsilon)}(u) du$$

$$-\varepsilon^2 \int_0^t \sigma^{(\varepsilon)}(u) \bar{\sigma}^{'} \gamma_{fj}(u) f_{fj}^{(\varepsilon)}(u) du + \varepsilon \int_0^t f_{fj}^{(\varepsilon)}(u) \gamma_{fj}^{'}(u) dW_u, \tag{31}$$

where $g_{dj}^{0,(\varepsilon)}(t) := \frac{-\tau_j f_{dj}^{(\varepsilon)}(t)}{1+\tau_j f_{dj}^{(\varepsilon)}(t)} \gamma_{dj}(t), g_{fj}^{0,(\varepsilon)}(t) := \frac{-\tau_j f_{fj}^{(\varepsilon)}(t)}{1+\tau_j f_{fj}^{(\varepsilon)}(t)} \gamma_{fj}(t);$ $x^{'}$ denotes the transpose of x and W is a r-dimensional standard Wiener process under the domestic terminal measure P; $\gamma_{dj}(s)$, $\gamma_{fj}(s)$ are r-dimensional vector-valued functions of time-parameter s; $\bar{\sigma}$ denotes a r-dimensional constant vector satisfying $\|\bar{\sigma}\| = 1$ and $\sigma^{(\varepsilon)}(t)$, the volatility of the spot exchange rate, is specified to follow a $\mathbf{R}_{++}$-valued general time-inhomogeneous Markovian process as follows:

$$\sigma^{(\varepsilon)}(t) = \sigma(0) + \int_0^t \mu(u, \sigma^{(\varepsilon)}(u)) du + \varepsilon^2 \sum_{j=1}^{N} \int_0^t g_{dj}^{0,(\varepsilon)}(u)^{'} \omega(u, \sigma^{(\varepsilon)}(u)) du + \varepsilon \int_0^t \omega^{'}(u, \sigma^{(\varepsilon)}(u)) dW_u, \tag{32}$$

where $\mu(s, x)$ and $\omega(s, x)$ are functions of s and x.

Finally, we consider the process of the Forex forward $F_{N+1}(t)$. Since $F_{N+1}(t) \equiv F_{T_{N+1}}(t)$ can be expressed as $F_{N+1}(t) = S(t) \frac{P_f(t, T_{N+1})}{P_d(t, T_{N+1})}$, we easily notice that it is a martingale under the domestic terminal measure. In particular, it satisfies the following stochastic differential equation

$$F_{N+1}^{(\varepsilon)}(t) = F_{N+1}(0) + \varepsilon \int_0^t \sigma_F^{(\varepsilon)}(u)^{'} F_{N+1}^{(\varepsilon)}(u) dW_u \tag{33}$$

where $\sigma_F^{(\varepsilon)}(t) := \sum_{j=0}^{N} \left(g_{fj}^{0,(\varepsilon)}(t) - g_{dj}^{0,(\varepsilon)}(t) \right) + \sigma^{(\varepsilon)}(t).$

Numerical Examples

We here specify our model and parameters, and confirm the effectiveness of our method in this cross-currency framework. First, the processes of domestic and foreign forward interest rates and of the volatility of the spot exchange rate are specified. We suppose $r = 4$, that is the dimension of a Brownian motion is set to be four; it represents the uncertainty of domestic and foreign interest rates, the spot exchange rate, and its volatility. Note that in this framework correlations among all factors are allowed. We also suppose $S(0) = 100$.

Next, we specify a volatility process of the spot exchange rate in (32) with

$$\begin{cases} \mu(s, x) = \kappa(\theta - x), (34) \\ \quad \omega(s, x) = \omega\sqrt{x}, \end{cases} \tag{35}$$

where θ and κ represent the level and speed of its mean-reversion respectively, and ω denotes a volatility vector on the volatility. In this section the parameters are set as follows; $\varepsilon = 1$, $\sigma(0) = \theta = 0.1$, and $\kappa = 0.1$; $\omega = \omega^* \bar{v}$ where $\omega^* = 0.3$ and $\bar{v}$ denotes a four dimensional constant vector given below.

We further suppose that initial term structures of domestic and foreign forward interest rates are flat, and their volatilities also have flat structures and are constant over time: that is, for all $j, f_{dj}(0) = f_d$, $f_{fj}(0) = f_f$, $\gamma_{dj}(t) = \gamma_d^* \overline{\gamma}_d 1_{\{t < T_j\}}(t)$ and $\gamma_{fj}(t) = \gamma_f^* \overline{\gamma}_f 1_{\{t < T_j\}}(t)$. Here, γ_d^* and γ_f^* are constant scalars, and $\overline{\gamma}_d$ and $\overline{\gamma}_f$ denote four dimensional constant vectors. Moreover, given a correlation matrix $\underline{C}$ among all four factors, the constant vectors $\overline{\gamma}_d$, $\overline{\gamma}_f$, $\overline{\sigma}$ and $\overline{v}$ can be determined to satisfy $\| \overline{\gamma}_d \| = \| \overline{\gamma}_f \| = \| \overline{\sigma} \| = \| \overline{v} \| = 1$ and $V'V = \underline{C}$ where $V := (\overline{\gamma}_d, \overline{\gamma}_f, \overline{\sigma}, \overline{v})$.

In this subsection, we consider four different cases for f_d, γ_d^*, f_f and γ_f^* as in Table 1. For correlations, the parameters are set as follows. The correlation between interest rates and the spot exchange rate are allowed while there are no correlations among the others. The correlation between domestic interest rates and the spot forex is 0.5 ($\overline{\gamma}_d'\overline{\sigma} = 0.5$) and the correlation between foreign interest rates and the spot forex is -0.5 ($\overline{\gamma}_f'\overline{\sigma} = -0.5$). It is well known that (both exact and approximate) evaluation of the long-term options is a difficult task with such complex structures of correlations.

Table 1: Initial Domestic/Foreign Forward Interest Rates and Their Volatilities

	f_d	γ_d^*	f_f	γ_f^*
case (i)	0.05	0.12	0.05	0.12
case (ii)	0.02	0.3	0.05	0.12
case (iii)	0.05	0.12	0.02	0.3
case (iv)	0.02	0.3	0.02	0.3

This table shows the initial term stractures of domestic and foreign forward interest rates and those of their volatilities, which are assumed to be flat. The figures in the first and second column are the initial value of the domestic interest rates and their volatility. The figures in the third and fourth columun are thoes of foregin interest rates.

Lastly, we make an assumption that $\gamma_{dn(t)-1}(t)$ and $\gamma_{fn(t)-1}(t)$, volatilities of the domestic and foreign interest rates applied to the period from t to the next fixing date $T_{n(t)}$, are equal to be zero for arbitrary $t \in [t, T_{n(t)}]$.

In Figure 1, we compare our estimations of the values of call and put options whose maturities are from ten to twenty years by an asymptotic expansion up to the fourth order to the benchmarks estimated by 10^6 trials of Monte Carlo simulation. In the simulation, we discretized the underlying processes by a Euler-Maruyama scheme with time step 0.05 applied the Antithetic Variable Method. For the moneynesses (defined by $K / F_{N+1}(0)$) less than one, the prices of put options are shown; otherwise, the prices of call options are displayed.

As seen in this figure, in general the estimators seems more accurate as the order of the expansion increases. Especially, for the deep out of the money put options the fourth order approximation performs much better and is more stable than the lower order approximation.

CONCLUSIONS

In this paper, we reviewed the general procedures for the explicit computation of the asymptotic expansion method. One procedure is that of conditional expectations based on the approach for iterated Ito integrals. The other is the alternative but equivalent calculation algorithm which computes the unconditional expectations directly instead of using conditional expectations.

For simplicity and space limitation, we focused on the simple case of Black-Scholes-type economy which illustrated our key ideas. Moreover, we applied the methods to option pricing in the cross currency Libor market model with a stochastic volatility of the spot exchange rate to illustrate the usefulness and accuracy of our approximation with high order expansions. In this practically important example, satisfactory results were confirmed even for options with a twenty-year maturity.

In this paper considers only path-independent European derivatives without considering jumps. Future research will develop a similar result in the presence of a jump component. Future research might also pursue an efficient method for the evaluation of multi-factor path-dependent or/and American derivatives.

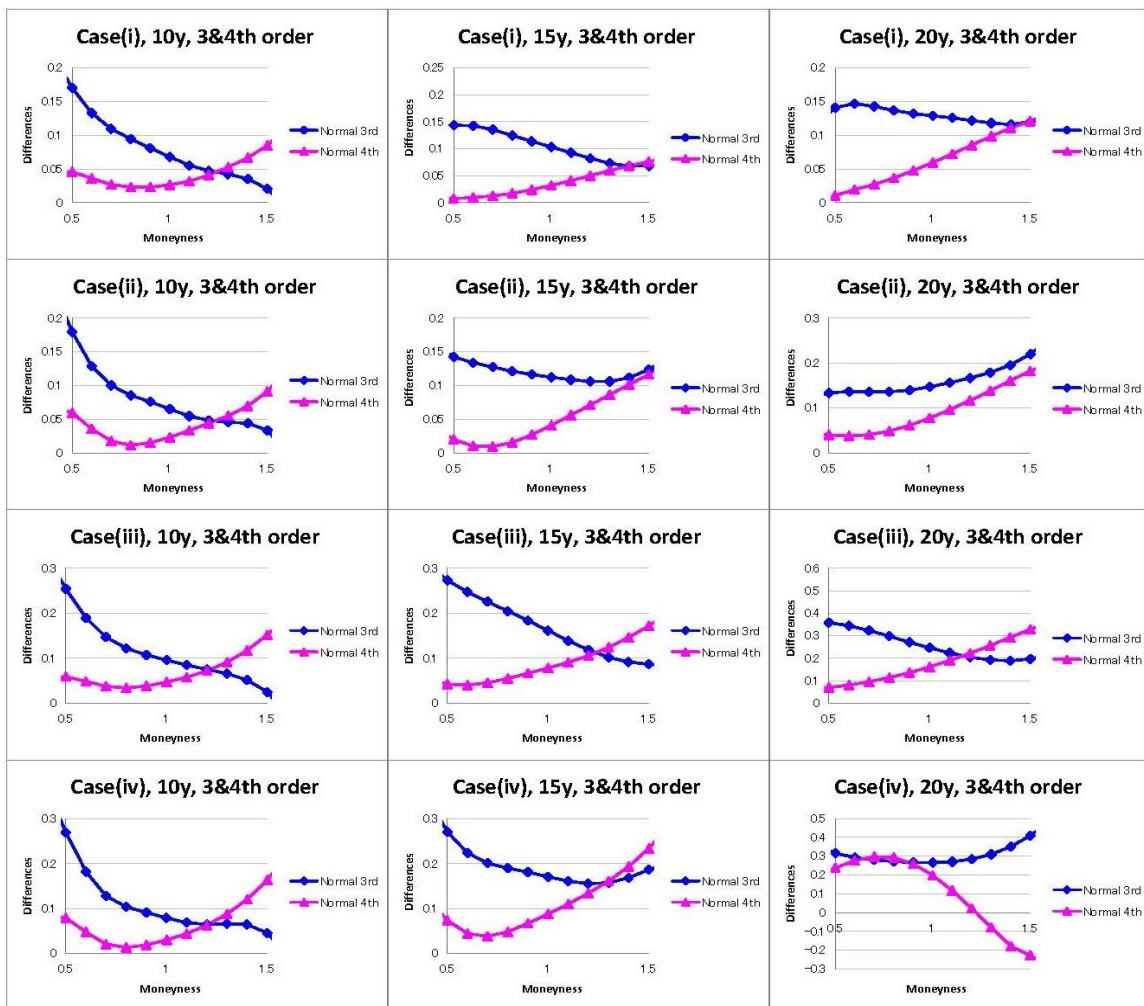Figure 1: Comparison of the Estimators by the Asymptotic Expansion and Simulations

Figure 1

This figure shows the differences between our estimators of option prices by the asymptotic expansion up to the third(blue lines) and fourth order(pink lines) and those by Monte Carlo simulations. The defferences are defined by (the estimate by the asymptotic expansion – that by simulation). "Moneyness" is defined by (Strike Rate / Spot Rate).

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