A COMPARISON OF DELTA HEDGING UNDER TWO PRICE DISTRIBUTION ASSUMPTIONS BY LIKELIHOOD RATIO

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ABSTRACT

This paper compares net profits from delta hedging through the Delta of a European call option, by assuming underlying stock prices follows a geometric Brownian motion (GBM) or a Variance-Gamma (VG) process. We employ the maximum likelihood estimation method to estimate corresponding parameters for each process. A Monte Carlo simulation is conducted to simulate spot prices and option prices and a likelihood ratio (LR) method is used to estimate the Delta of the call option over different sample paths. We then implement a dynamic delta hedging strategy through the simulated spot prices, option prices and Delta at different hedging frequencies. Finally, we compare net profits calculated from hedging corresponding to a GBM or a VG process.

JEL: G13, G15, G17

KEYWORDS: likelihood ratio, Variance-Gamma, geometric Brownian motion, delta hedging

INTRODUCTION

Delta hedging is a particular type of hedging strategy. The fundamental of delta hedging is to adjust the shares of stocks longed or shorted according to changes of option prices. Delta is defined as the rate of changes of option prices to spot prices. Therefore, Delta plays an important role in hedging strategy, since it measures the sensitivity of option prices to spot prices and determines how many shares of stocks to purchase or sell to offset risks from changes of option prices. The gradient estimation technique has been widely applied to estimate Delta. Two widely-used gradient estimation methods are (i) the likelihood ratio (LR) method, and (ii) the infinitesimal perturbation analysis (IPA) method.

The main purpose of this paper is to compare net profits from delta hedging by assuming underlying stock prices follow a geometric Brownian motion (GBM) or Variance-Gamma (VG) process. Following Jarrow and Turnbull (1999), we employ the dynamic hedging strategy to hedge periodically before a European call option matures. Since Delta changes frequently before maturity, we estimate the Delta each time we intend to delta hedge, in order to improve the accuracy of stock shares required to offset the risk.

The remainder of this paper is organized as follows. We first provide a literature review of delta hedging, gradient estimation technique, as well as a geometric Brownian motion process and a Variance Gamma process. Then, we introduce the delta hedging strategy, as well as the background of GBM and VG processes. We also provide details of how to estimate Delta for the two processes by the LR method. Finally, a numerical experiment of dynamic delta hedging is conducted to compare net profits from GBM and VG processes.

LITERATURE REVIEW

Delta hedging has been widely applied by investors who are long or short options to hedge risks from changes of option prices. Due to its broad application in financial engineering, there is a vast literature on delta hedging. Hull (2003) provides a general introduction of hedging strategies including delta hedging. Jarrow and Turnbull (1999) provide a detailed explanation of how to implement dynamic delta hedging and replicate portfolios to achieve a delta-neutral position.

The gradient technique is one area in the class of Monte Carlo simulation, which has been broadly applied in financial engineering and has been studied and summarized in Glasserman (2004). Fu (2006) reviews kinds of methods of gradient estimation and their applications in the finance community. Fu and Hu (1995) first bring the gradient estimation technique IPA method into option pricing and sensitivity analysis of options. Broadie and Glasserman (1996) then employ IPA and LR methods to price European and Asian options and analyze sensitivities of these two options. Fu (2008) reviews techniques and applications to derivatives securities. Cao and Guo (2011-1) assume stock prices follow a Variance-Gamma process, and employ the forward difference, IPA and LR method to estimate Greeks for a European call option. Cao and Guo (2011-2) compare the gradient estimates from the Variance Gamma model assumption and geometric Brownian motion model assumption. Cao and Guo (2011-3) analyze the statistic properties of net profits from delta hedging via deltas estimated from LR and IPA methods under a geometric Brownian motion model. Cao and Guo (2011-4) compares results of delta hedging through deltas calculated from two price distributions (a GBM and a VG) by IPA method. In this paper we employ the LR method to estimate Deltas which play an important role in delta hedging, due to its popularity in empirical research.

Before implementing the LR method, we need to make certain assumptions on the underlying processes. The popularly used geometric Brownian motion model is also called the Black-Scholes model, which was first proposed by Black and Scholes (1973) and Merton (1973) assumes stock prices follow a geometric Brownian motion process. However, empirical evidence suggests that the GBM has some imperfections and does not describe the statistical properties of financial time series well. The Variance Gamma (VG) process, as one of the most popular Levy process, was first introduced to the literature in Madan and Seneta (1990) then applied to option pricing in Madan and Milne (1991). Madan, Carr and Chang (1998) developed the VG process by adding one more parameter to describe the negative skewness of stock prices in the market. This VG process has shown more accuracy in pricing stocks. Fu (2007) reviews how to apply this model by Monte Carlo simulation to price options and other derivatives. Cao and Fu (2010) estimate the Greeks of a basket of options called Mountain Range options in the assumption that each asset is defined by this model.

DELTA HEDGING STRATEGY

Delta is the rate of changes of option prices with respect to price changes of underlying assets. In other words, Delta measures the sensitivity of a derivative f say options or the portfolios of options, with respect to stock prices S. We could define Delta Δ as :

$$\Delta = \frac{\partial f}{\partial S} \tag{1}$$

Equation (1) implies that when stock prices change by a small amount ΔS, option prices would change correspondingly by an amount of $\Delta \times \Delta S$. An investor could hedge the risk by adjusting (purchase or sell) shares of stocks to make the portfolio's delta be zero, also called the delta-neutral portfolio. Delta hedging is a trading strategy which attempts to maintain a delta-neutral portfolio dynamically by offsetting the change of option positions through the change of stock positions. As Delta changes, an investor's risk-neutral position (delta-hedged position) would exist for only a short time. Thus, we need to adjust hedging positions periodically, which is called rebalancing. If we could rebalance immediately when stock prices change, perfect hedge is achieved; however, perfect hedge is always difficult to achieve in the real world. What we can do is to employ a dynamic hedging strategy and make hedging periods as short as possible. Next, we explain the procedure of delta hedging by setting the portfolio's delta zero.

Suppose an investor writes N_0 number of European call options which will mature after a period of $\tilde{T}$, and each option covers 100 shares of stocks. The investor can buy $100 \times N_0$ shares of stocks to hedge his position, since the gain or loss on his option position can be offset by the loss or gain on his stock position. However, as time changes, Delta changes; the risk-neutral position is destroyed. Thus, he has to adjust the portfolio by delta hedging every Δt period, i.e., at $t = 0, t_1 = \Delta t, \cdots, t_{k-1} = (k-1)\Delta t$, where k is the largest integer satisfying $t_{k-1} < \tilde{T}$ and $t_k > \tilde{T}$. The purpose of delta hedging is to make the value of a portfolio insensitive to small changes of option prices to maintain a delta-neutral portfolio position. In addition, the portfolio is self-financing. The options an investor writes would cover a total of $N = 100 \times N_0$ shares of stocks. Assume he will long m_0 shares of the stock and borrow B_0 dollars at t_0 to offset the risk. Denote the option price by f_0, the stock price by S_0 and the Delta Δ_0 at t_0. The value of the portfolio P at t_0 is set to be 0, that is:

$$P = -N \times f_0 + m_0 \times S_0 + B_0 = 0 . \tag{2}$$

Taking the derivative of the value of the portfolio P with respect to the stock price at t_0, we have:

$$\frac{dP}{dS_0} = -N \times \Delta_0 + m_0,$$

which could be considered as the delta of the portfolio Δ_p, i.e., $\Delta_p = \frac{dP}{dS_0}$. In order to maintain the $\Delta_p = \frac{dP}{dS_0}$ portfolio delta-neutral, we have $\Delta_p = 0$, i.e.,

$$\Delta_p = -N \times \Delta_0 + m_0 = 0. \tag{3}$$

From Equation (2) and Equation (3), we could calculate shares of stocks to purchase (m_0) and the amount of dollars to borrow (B_0) at t_0. As is mentioned earlier, we have to rebalance the portfolio periodically. At the second hedging period t_1, we have delta Δ_1 at t_1 and follow the same procedure above by setting both the net value and delta of the portfolio to zero, to get the shares of stocks to purchase (m_1) and the amount of money to borrow (B_1) at t_1. We need to rebalance k times from $t = 0$ to $t = t_{k-1}$ before the option matures at $\tilde{T}$.

Following the procedures of rebalancing described above, we have a total of $M_s = \sum_{i=0}^{k} m_i$ shares of stocks at $\tilde{T}$. In order to calculate the replication cost at $\tilde{T}$ in each sample path, we need to consider (i) the payoff V_1 from selling M_s shares of stocks considering having written N_0 number of European call options, and (ii) the cumulative cost V_2 which is accumulated with interest form borrowing cash during the whole hedging procedure. V_1 can be calculated as:

$$V_1 = M_s \times \max(S_{T,} K), \tag{4}$$

where K is the strike price of an option. It is noteworthy that generally $M_s < N$. Let C_i be the cumulative cost of cash borrowed till t_i and B_i be the amount of cash borrowed at t_i we have

$$C_0 = B_0,$$
$$C_1 = B_1 + C_0 \times \exp(r \times \frac{\Delta t}{365})$$
$$\cdots$$

$$C_{k-1} = B_{k-1} + C_{k-2} \times \exp(r \times \frac{\Delta t}{365})$$

V_2 can be calculated as:

$$V_2 = C_{k-1} \times \exp(r \times \frac{\tilde{T} - t_{k-1}}{365}). \quad (5)$$

Therefore, the replication cost V_3 is

$$V_3 = V_2 - V_1.$$

To calculate the net gain of delta hedging in different sample paths, we need to calculate the payoff V_4 from selling all the call options the investor writes at $t = 0$. Thus, V_4 is

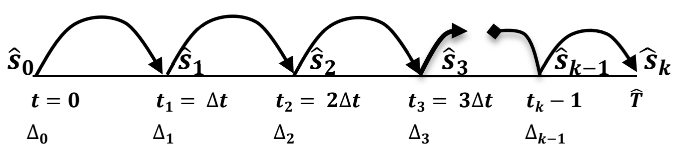
$$V_4 = f_0 \times N_0$$
$$V_2 = f_0 \times N_0 \times \exp\left(r \times \frac{\tilde{T}}{365}\right).$$

The net gain of delta hedging periodically in one sample path is

$$V_{net} = V_4 - V_3.$$

We shall explain how to delta hedge in one sample path in Figure 1.

Figure 1: A Sample Path of Delta Hedging



Notes: This figure shows one sample path of estimating deltas and when to delta hedge.

ALGORITHMS OF DYNAMIC DELTA HEDGING

Algorithm 1 for Spot Prices:

First, we need to simulate all spot prices $\hat{S}_i$ at different periods t_i. The algorithm to simulate spot prices in one sample path is as follows:

At $t_0 = 0$, the spot price is $\hat{S}_0 = \tilde{S}_0$.

At $t_1 = \Delta t$, the spot price $\hat{S}_1$ can be calculated through (10) or (15) by setting$t = \Delta t$, $S_0 = \hat{S}_0$ and $S_t = \hat{S}_1$.

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At $t_{k-1} = (k-1)\Delta t$, the spot price $\hat{S}_{k-1}$ can be calculated through (10) or (15) by setting $t = \Delta t$, $S_0 = \hat{S}_{k-2}$ and $S_t = \hat{S}_{k-1}$.

At $t_k = \tilde{T}$, the spot price $\hat{S}_k$ can be calculated through (10) or (15) by setting $t = \tilde{T} - (k-1)\Delta t$, $S_0 = \hat{S}_{k-1}$ and $S_t = \hat{S}_k$.

Next we show Algorithm 2 for the calculation of Delta. We need to simulate the Delta at hedging periods t_1. The algorithm to estimate the Delta in one sample path is as follows:

At $t_0 = 0$ the estimators for Delta can be calculated through (12) or (17) by setting $S_0 = \hat{S}_0$, and $T = \tilde{T} - t_0 = \tilde{T}$.

At $t_1 = \Delta t$, the estimators for Delta can be calculated through (12) or (17) by setting $S_0 = \hat{S}_1$, and $T = \tilde{T} - t_1 = \tilde{T} - \Delta t$.
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At $t_{k-1} = (k-1)\Delta t$ the estimators for Delta can be calculated through (12) or (17) by setting $S_0 = \hat{S}_{k-1}$, and $T = \tilde{T} - t_{k-1} = \tilde{T} - (k-1)\Delta t$.

At $t_k = \tilde{T}$, we cannot hedge on the maturity day and thus do not need to estimate the Delta.

To employ the hedging strategy shown above, we need to estimate the Delta first. In this paper, we estimate the Delta from a GBM or a VG by LR, respectively. In the following sections, we provide an introduction of the LR method in the gradient estimation technique as well as the GBM and the VG processes. In addition, estimators of the Delta by LR are also provided.

GRADIENT ESTIMATION TECHNIQUE: LIKELIHOOD RATIO METHOD

Simulation and gradient estimation are very useful in financial engineering applications. To employ delta hedging, we first estimate the Delta. Delta can be calculated by taking the derivative of the option prices with respect to spot prices. Let's set up the problem first. Assuming the objective function $V(\xi)$ depends on the parameter ξ, we focus on calculating:

$$\frac{d\,V(\xi)}{d\xi}.$$

Suppose the objective function is an expectation of the sample performance measure, that is:

$$V(\xi) = E[L(\xi)] = E[L(X_{1,}X_2, \cdots, X_n; \xi)] \tag{6}$$

where $X = X_{1,}X_2, \cdots, X_n$ are dependent on ξ, and n is a fixed number of random variables. Using the law of unconscious statistician, the expectation can be written as:

$$E[L(X)] = \int y dF_L\,(y), \tag{7}$$

where F_L is the distribution of L; and

$$E[L(X)] = \int L(x) dF_x\,(x), \tag{8}$$

where F_x is the distribution of an input random variable X. According to different ways of writing $V(\xi)$ above, we have several methods to estimate the gradient of $V(\xi)$, i.e., finite difference, IPA and LR, of which the last two belong to indirect methods.
To make sense of the right hand side of Equation (11), we write the expectation of $L(X)$ as:

$$E[L(X)] = \int L(x) f_X(x, \xi) dx, \tag{9}$$

where f_X is the probability density function of X. The dependence of parameter ξ can be path-wise from the input random variable X as shown in Equation (8), and LR method originally comes from taking the derivative of Equation (9). Assume the probability density function f_X of X is differentiable. The LR estimate is:

$$\frac{dE[L(X)]}{d\xi} = \int_{-\infty}^{+\infty} L(x) \frac{df_X(x,\xi)}{d\xi} dx = \int_{-\infty}^{+\infty} L(x) \frac{dlnf_X(x,\xi)}{d\xi} f_X(x)\, dx,$$

and the estimator is

$$L(x) \frac{dlnf_X(x,\xi)}{d\xi},$$

where $\frac{dlnf_X(x,\xi)}{d\xi}$ is the score function.

A European call option gives the buyer the right, not the obligation to buy a certain amount of financial instrument from the seller at maturity for a certain strike price. Let S_t be a stock price, T be the maturity time, K be the strike price, and r be the risk-free interest rate. The price (value) of the European call option at t is

$$V_T = e^{-rT}(S_T - K)^+,$$

where S_T can follow a GBM process or a VG (GVG or DVG) process.

ESTIMATING UNDER GEOMETRIC BROWNIAN MOTION PROCESS

A stochastic process price S_t follows a geometric Brownian motion if price $\log(S_t)$ is a Brownian motion with initial value price $\log(S_0)$. In the Black-Scholes model, the price of an underlying stock S_t following a geometric Brownian motion process satisfies

$$\frac{dS_t}{S_t} = \mu dt + \tilde{\sigma} dW_{\mathrm{t}},$$

where W_{t} is a standard Brownian motion. With dividend yield q, spot S_0, volatility $\tilde{\sigma}$ and drift $\mu = r - q$, we can obtain the stock price:

$$S_t = S_0 \exp\big((r - q - \tilde{\sigma}^2)t + \tilde{\sigma} W_{\mathrm{t}}\big).$$

The stock price S_t can be simulated through

$$S_t = S_0 \exp\Big((r - q - \tilde{\sigma}^2)t + \tilde{\sigma}\sqrt{t}\,\tilde{Z}\Big) \tag{10}$$

where $\tilde{Z}$ represents a standard normal random variable. The density of stock price S_t is

$$f(x) = \frac{1}{x\tilde{\sigma}\sqrt{2\pi t}} exp\left\{-\frac{1}{2}\left[\frac{1}{\tilde{\sigma}\sqrt{t}}\left(ln\frac{x}{S_0} - \left(r - \tilde{\sigma} - \frac{\tilde{\sigma}^2}{2}\right)t\right)\right]^2\right\} \tag{11}$$

Applying the density function in Equation (11), we have

$$\frac{dE[V_T]}{d\,S_0} = \int_0^\infty e^{-rT}\,(x-K)^+ \times \frac{\mathrm{d}f(x)}{d\,S_0}dx = \int_0^\infty e^{-rT}\,(x-K)^+ \times \frac{d(ln\mathrm{f}(x))}{d\,S_0}\mathrm{f}(x)dx$$

Therefore, the estimator of Delta for a European call option through LR is

$$e^{-rT}\,(x-K)^+ \times \frac{d(ln\mathrm{f}(x))}{d\,S_0}. \tag{12}$$

ESTIMATING UNDER VARIANCE GAMMA PROCESS

The Variance Gamma Process is a Levy process of independent and stationary increments. The characteristic function of $VG(\sigma,\nu,\theta,t)$ is given by

$$\Phi_{VG}(\mu,\sigma,\nu,\theta,t) = (1 - iu\theta\nu + \tfrac{1}{2}\sigma^2\nu u^2)^{-t/\nu}.$$

There are two ways to define the VG process. The VG process can be defined as a Gamma-time-changed Brownian motion subordinated by a gamma process. Let W_t be a standard Brownian motion, $B_t^{(\mu,\sigma)} = \mu t + \sigma W_t$ be a Brownian motion with a constant drift rate μ and volatility σ, $\gamma_t^{(\nu)}$ be a gamma process with drift $\mu = 1$ and variance parameter ν. The representation of VG process (say GVG) is:

$$X_t = B_{\gamma_t^{(\nu)}}^{(\theta,\sigma)} = \theta\gamma_t^{(\nu)} + \sigma W_{\gamma_t^{(\nu)}} \tag{13}$$

Second, the VG process is the difference of two gamma processes. Let $\gamma_t^{(\mu,\nu)}$ be the gamma process with drift parameter μ and variance parameter ν, the representation of the VG process as the difference of gamma process is:

$$X_t = \gamma_t^{(\mu_+,\nu_+)} - \gamma_t^{(\mu_-,\nu_-)}, \tag{14}$$

where $\mu_\pm = (\sqrt{\theta^2 + 2\frac{\sigma^2}{\nu}} \pm \theta)/2$, and $\nu_\pm = (\mu_\pm)^2\nu$.

Under the risk-neutral measure, with no dividends and a constant risk-free interest rate r, a stock price is given by

$$S_t = S_0 \exp\big((r+\omega)t + X_t\big), \tag{15}$$

Where $\omega = \ln\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right)/\nu$ is the parameter that makes the discounted asset price a martingale.

Madan, Carr and Chang (1998) propose that the density function of the log-price $Z = \ln(S_t/S_0)$ is

$$h(z) = \frac{2\exp\left(\frac{\theta x}{\sigma^2}\right)}{\nu^{\frac{t}{\nu}}\sqrt{2\pi}\sigma\,\Gamma\left(\frac{t}{\nu}\right)}\left(\frac{x^2}{\frac{2\sigma^2}{\nu}+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}}\widehat{K}\left(\frac{1}{\sigma^2}\sqrt{x^2(\frac{2\sigma^2}{\nu}+\theta^2)}\right), \tag{16}$$

where $\widehat{K}$ is the modified Bessel function of the 2nd kind, and is $x = z - rt - \frac{t}{\nu}\ln\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right)$. Since $h(z)$ doesn't contain S_0, we have to use the Jacobian transform to get the density function of S_T to calculate the derivative with respect to S_0:

$f_{S_T}(s) \times \left|\frac{\partial S_T}{\partial Z}\right| = h(z).$

Therefore, we can get the density function of S_T:

$f_{S_T}(s) = h(lns - ln\, S_0) \times \frac{1}{s}.$

To calculate the Delta, we use $f_{S_T}(s)$ to apply the LR. Since

$\frac{dE[\,V_T]}{d\,S_0} = \int_0^\infty e^{-rT}\,(s-K)^+ \times \frac{d(lnf_{S_T}(s))}{d\,S_0} f_{S_T}(s)ds.$

the estimator of the Delta from LR under VG is:

$$e^{-rT}\,(s-K)^+ \times \frac{d(lnf_{S_T}(s))}{d\,S_0}. \tag{17}$$

When the stock price follows a GVG or a DVG process, the estimator in (17) would be the estimators for GVG or DVG accordingly.

NUMERICAL EXPERIMENT

In this paper, we analyze historical data in WRDS of the stock prices of Google Ltd. from March 10th, 2008 to September 10th, 2008. Assuming a stock price follows a geometric Brownian motion or a Variance Gamma process, we apply the MLE method to estimate the corresponding parameters. Assuming the maturity time for the option is 30 days, i.e., $\tilde{T} = 30/365$, the risk free interest rate minus the dividend rate is $r - q = 0.0245174$, we get the variance parameter $\tilde{\sigma} = 0.28983965441613$ for the GBM process; and $\sigma = 0.21370332702956, \nu = 0.01879357471038$, and $\theta = -0.19286112983688$ for the VG process. The spot price is $\tilde{S}_0 = 433.75$ at $t = t_0$ and the strike price is $K = 440$.

We apply a Monte Carlo simulation to follow the algorithm for spot prices and algorithm for the Delta described above to simulate for 10000 sample paths. Moreover, after having the corresponding spot prices and the Delta on each sample path, we employ delta hedging technique to calculate the net gains on each sample path. The summary statistics for the net profits from delta hedging only once initially, i.e., $\Delta t = \tilde{T} = 30/365$ by the methods above is shown in Table (1).

CONCLUSION

Assuming stock prices follow two price distributions, a geometric Brownian motion process and a Variance Gamma process, we employ the dynamic delta hedging strategy to identify net profits and analyze the statistical properties under these two assumptions. Deltas play an important role in the hedging strategy. For different hedging times, we download the historical data of the stock prices of Google Ltd. and calculate the corresponding deltas through one of the gradient estimation techniques called likelihood ratio method. A comparison is made on the numerical results obtained above.

Our main findings can be summarized as follows: 1) The mean values of net gain from higher hedging frequency are always bigger than the ones from lower hedging frequency. But in our experiment, the mean values of hedging just once initially are higher than ones from higher frequency. This exception happens is because the standard error is very large, and the results are probably biased. 2) The standard errors of the net profit from higher frequency are always lower than the ones from the net profit from lower frequency. 3) The mean value of net gain following the Variance Gamma process is bigger than the mean following the geometric Brownian motion process. 4) The mean value of net gain from GVG and DVG are close. Further work is needed to reduce the big standard errors of net profits from hedging at low frequency, making the all results unbiased.

Table 1: Summary Statistics of Net Profit by Delta Hedging

Hedge once initially	**GBM LR**	**GVG LR**	**DVG LR**
Mean	4484.4	4492.4	4439.3
StdErr	100.26	99.85	101.47
Hedge every 14 days			
Mean	3895	3981	3904
StdErr	65.14	60.33	59.73
Hedge every 7 days			
Mean	4059	4051	4077
StdErr	53.52	50.73	69.04
Hedge every 3 days			
Mean	4583	4607	4554
StdErr	26.41	20.98	20.58

Notes: This table shows the summary statistics of net profit by delta hedging at different hedging frequencies. Mean denotes the mean value of net profit, while StdErr is the standard error of net profit. GBM LR is the results from delta hedging with respect to a GBM process by LR method. GVG LR is the results from delta hedging with respect to a GVG process by LR method. DVG LR is the results from delta hedging with respect to the GBM process by LR method. Panel A shows the results of summary statistics of net profits by delta hedging once initially, i.e. $\Delta t = \tilde{T} = \frac{30}{365}$*. Panel B shows the results of summary statistics of net profits by delta hedging every 14 days , i.e.,* $\Delta t = \tilde{T} = 14/365$*. Panel C shows the results of summary statistics of net profits by delta hedging every 7 days, i.e. ,*$\Delta t = \tilde{T} = 7/365$*. Panel D shows the results of summary statistics of net profits by delta hedging every 3 days, i.e.,* $\Delta t = \tilde{T} = 3/365$*.*

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