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ESTIMATION OF PORTFOLIO RETURN AND VALUE AT RISK USING A CLASS OF GAUSSIAN MIXTURE DISTRIBUTIONS

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ABSTRACT

This paper deals with the estimation of portfolio returns and Value at Risk (VaR), by using a class of Gaussian mixture distributions. Asset return distributions are frequently assumed to follow a normal or lognormal distribution. It also can follow Brownian motion or Geometric Brownian motion based upon the Gaussian process. However, many empirical studies have shown that return distributions are usually not normal. They often find evidence of non-normality, such as heavy tails, excess kurtosis, finite moments, etc. We propose a class of Gaussian mixture distributions to approximate the return distributions of assets. This class of Gaussian mixture distributions, having good statistical properties, can accurately capture the above-mentioned statistical characteristics of return distributions. The model is applied easily to estimate the return distribution of a portfolio, and to evaluate the VaR. We demonstrate the model theoretically and provide some applications.

JEL: G10, G11, G32

KEYWORDS: Gaussian mixture distribution, convolution density, portfolio, Value at Risk

INTRODUCTION

In recent times, many works have focused on modeling asset return distributions, by assuming returns follow a Brownian motion or Geometric Brownian motion. It is therefore a Gaussian process with a time factor in its mean and variance or it follows a normal or lognormal distribution. In practice, the normality assumption of returns are usually rejected by statistical tests, such as the Jarque-Bera test (Jarque and Bera 1980), based on the kurtosis and the skewness of observed data. Due to non-normal evidence, such as heavy tails, excess kurtosis (Carol, A., 2004), some researchers simply assume that a financial return follows a distribution with heavy tails, such as the Student *t* distribution, a distribution derived from the Pearson VII family, or a Generalized Error Distribution (GED). However, it is difficult to use a single distribution family to approximate the return distribution with various distributional characteristics (McLachlan and Peel 2000, Tan 2005, Tan and Tokinaga 2006, Tan 2007a). In this paper, we propose a class of Gaussian mixture distributions to approximate the return distribution of an asset or portfolio. In our application, we also show how to evaluate the VaR using our proposed method.

The remainder of the paper is organized as follows. Next the relevant literature is presented. This is followed by a discusses of conventional distribution assumptions. Next the theoretical framework and some applications of our model are presented. The paper closes with some concluding remarks.

LITERATURE REVIEW

Many studies on modeling return distributions of financial assets have been conducted. Among them, the most used three-type distributions are the normal, the lognormal and the non-Gaussian stable distributions. Other types of distributions, such as the Student t, the skewed Student t, the generalized t, the Generalized Error Distribution (GED), the skewed GED, and mixture distribution of Gaussian distributions have been examined and proposed.

The normal distribution is one of the most commonly observed and applied distributions. It was widely used in the 1700's and successfully applied to astronomical data analysis by Karl Gauss in 1800, and became known as the Gaussian distribution. From the late 1960s, the empirical analyses failed to support the normal assumption on estimating the return distribution of real financial data. For example, Mandelbrot (1963) claimed that while financial prices (or its logarithm) following a Brownian motion is mathematically convenient, it is hard to fit the real financial data with this assumption. Meanwhile, Fama (1965) analyzed equilibrium asset pricing and observed that the daily return distribution follows a non-Gaussian distribution. Furthermore, both Mandelbrot (1963) and Fama (1965) have pointed out that excess kurtosis and heavy tails exist in real financial data.

Many empirical studies reject normality of returns. For instance, both Hsu, et al. (1974) and Hagerman (1978) carried out empirical studies and concluded that return distributions are non-normal. Bollerslev (1987) found leptokurtosis in monthly Standard & Poor's 500 Index returns. Kariya, et al. (1995) and Nagahara (1996) find the return distribution of Japanese stocks are fat-tailed and skewed. Kitagawa, Sato and Nagahara (1999) found that daily or weekly return distributions are not normal but fat-tailed and skewed according to observed financial data. Harvey and Siddique (2000), as well as Premaratne and Bera (2000) confirmed the asymmetry of return distribution exists in real business data. Recently, Gerig, Vicente and Fuentes (2009) presented a model that explained the shape and scaling of the distribution of intraday stock price fluctuations and verified the model by using a large database made up of several stocks traded on the London Stock Exchange. Their results showed that the return distributions for these stocks are non-Gaussian, similar in shape and appear to be stable over intraday time scales.

Thus, normality is not acceptable as a rational assumption for returns. In line with the empirical analyses, some researchers found that return distributions have heavy tails and then simply assumed the financial returns follow the Student t distribution. Blattberg and Gonedes (1974) claimed that the Student t distribution is more suitable to estimate return distributions. On the other hand, Seong and Sang (2007) employed a skewed Student t on the estimation of Value-at-Risk, for long memory volatility processes in Japanese financial Markets. Kercheval and Wu (2010) applied the skewed Student t to portfolio optimization, because the skewed Student t can capture the characteristics of the skewness of observed empirical distributions well. Glasserman (2003) also confirmed the basic settings of return distributions are crucial based on numerical approaches. He concluded that a slightly different setting can lead to a completely different risk measurement, since it uses the variances and covariances between all the component asset risks, or the historical data based Monte Carlo simulations.

In order to model the heavy-tailed behavior, Akgiray and Geoffrey (1988) and Nolan (1997) proposed a non-Gaussian stable distribution to describe a return distribution. However, a shortcoming of such a model is that a non-Gaussian stable distribution does not have finite moments. The estimates of variance and kurtosis tend to be increasingly large and not to converge as the sample size increases.

Since the 1980s, the inconsistency between the theoretical models and empirical analysis for the observed skewness and excess kurtosis has been well discussed. To model these statistical properties, Jarrow and Rudd (1982) suggested using an Edgeworth series expansion as well as the Gram-Charlier series to approximate the real distribution of asset returns when the real distribution is unknown. This approach also has been adopted in option pricing by researchers, such as Knight and Satchell (1997), and Corrado and Sue (1997). Although, these expansions can be used to approximate distributions, they are not popularly applied in real data analysis because of the difficulties in mathematical calculation and the existence of non-convergence.

Other proposed distributions to incorporate the observed skewness and excess kurtosis in the financial markets are skewed generalized t distribution, the Generalized Error Distribution (GED), the skewed GED, etc. For example, Theodossiou (1998) suggested using a skewed generalized t distribution which includes the Student t and skewed Student t to model return distributions. Furthermore, Theodossiou (2000) pointed out that a skewed GED fits the financial data well, while the asymmetry and excess kurtosis are observed in the financial data.

It has been shown that it is difficult to simply use a single distribution family to approximate return distributions with various distributional characteristics (McLachlan and Peel, 2000, Carol, 2004). Moreover, Tan (2005), Tan and Tokinaga (2006, 2007a) pointed out that conventional assumptions are inconsistent with the empirical analysis, since either a single distributional family assumption can hardly catch excess kurtosis and heavy tails, and having finite moments simultaneously. It is more complicated in some multimodal cases. Thus, serious estimation bias could be introduced when these assumptions are applied to risk measurement and management risk.

The estimation of Value at Risk (VaR) advocated by Jorion (1996) as a risk assessment tool at financial institutions strongly relies on the shape of a return distribution. For a normal distribution assumption, the VaR such as 1%, or 5% can be easily estimated. While compared to the normal case, a distribution with heavy tails such as a Student t distribution would yield a different estimate at the same percentage level. Also as shown in Zangari (1996), the VaR would be underestimated using a normal assumption under the circumstances of heavy tail phenomena. It might lead to a hefty loss in capital management.

Tan and Tokinaga (2007a) studied the statistical properties of a Gaussian mixture distribution and found that it can provide an accurate approximation for a probability distribution function for data with a complicated empirical distributional shape, by catching heavy tails behavior and excess kurtosis, being finite moments, even for the multimodal cases. This approach has been applied in the RiskMetrics™ (Longerstaey and More, 1995) advocated by Morgan (1996), namely, a Gaussian mixture distribution is utilized to reveal the fat-tailed behavior.

Moreover, a fat-tailed distribution corresponds to a jump process. For example, a return process follows a Geometric Brownian motion with a jump factor. As pointed out in Tan (2005), Tan and Tokinaga (2006, 2007a), in practice, the statistical characteristics, such as, skewed distributional shape, or heavy-tailed behavior, is easily modeled (captured) using a class of Gaussian mixture distributions, as well as a multimodal distributional shape. A mixture distribution has the flexibility to approximate various shapes of distributions, by adjusting its component weights and other component distributional parameters, such as mean and variance.

Furthermore, an effective method to estimate the tail distribution related to the rare events (for example, estimation of the VaR) through a simulation approach, is to use the Importance Sampling (IS) method (Glasserman, 2003, Tan and Tokinaga, 2007b). The IS method can not only reduce the size of simulation samples, but also improve the accuracy of the estimated probabilities. However, for some distributions, it is difficult to identify the optimal parameter in the IS method. But, when the probability density distribution (p.d.f.) of the return distribution is a Gaussian mixture distribution, Tan and Tokinaga (2007b), Tan, et al. (2011) showed that finding the optimal parameter in the IS method is guaranteed. It can increasingly improve the effectiveness of simulation compared to the standard Monte Carlo simulation.

These works have shown that a class of Gaussian mixture distributions can capture the distributional characteristics of various distributions by scaling different component distributions to adjust its statistical properties to meet the observed data, such as the combinational weights, means and standard deviations in this mixture distribution class to fit the real data. Also a Gaussian mixture has advantage in estimating the rare events.

Thus, in this paper, we propose a Gaussian mixture distribution to approximate the return distribution of an asset. We then extend our results and further theoretically show some good statistical properties of this class of Gaussian mixture distributions when used to estimate the return distribution of a portfolio or the VaR (Jorion, 1996). It can provide an accurate distribution approximation and keep the model easily useable in both academic research and business practice.

CONVENTIONAL DISTRIBUTION ASSUMPTIONS

In this section, we review the merits and demerits of conventional distribution assumptions for returns. Hereafter, the normal distribution, the lognormal distribution and the non-Gaussian stable distributions are referred to as conventional distributions. Each of them has its own merits and demerits when applied to estimate an asset return.

The merit the normal distribution is that it makes the return easily tackled. However, the disadvantages are: The simple return has lower bound is -1, but there is no lower bound in the normal distribution. If one-period simple return is normal, then the multi-period simple return is not normal. Empirical results do not support normality since excess kurtosis and heavy tails are well observed in returns.

The Lognormal distribution has the following merits: There is no lower bound in such a setting. It allows the multi-period also to be normally distributed. However, the disadvantage is that it cannot capture the characteristics of excess kurtosis and tail behavior in returns. Therefore, empirical analyses do not support log-normality either.

The merit of the Stable Distribution is it allows the sum of returns still belong to the stable family. It can also fit the tail behavior and catch the excess kurtosis well in some cases. While, the problem is that the non-Gaussian stable distribution has infinite moments. The estimates of variance and kurtosis tend to be increasingly large and not to converge as the sample size increases.

Except the above three conventional distributions, other distributional assumptions of assets returns have been proposed such as the Student t, the skewed Student t, the generalized t, GED, the skewed GED, which turn out to be very complicated distributional forms when applied to estimate the return distribution of a portfolio. Therefore, it is difficult to use merely one of these distribution families to approximate an asset/portfolio return distribution. A Gaussian mixture distribution is assigned to a return distribution of each asset in a portfolio so that a good estimation of the return distribution of a portfolio can be realized. Furthermore, when this class of Gaussian mixture distributions is applied to estimate a portfolio, the convolution of such a class of Gaussian mixture distributions yields the same class of Gaussian mixture distributions with new distributional parameters, namely, the weights, means and variances, which makes the model easily tackled. Our theoretical framework is summarized as follows.

THEORETICAL FRAMEWORK

Suppose we have return distributions $f_1(x)$, $f_2(x)$, … $f_m(x)$ for each investing period *j* (if the total investing periods are *m*). Each $f_j(x)$ is denoted as a Gaussian mixture distribution. Each $f_j(x)$ is a Gaussian mixture distribution of a portfolio return for the *jth* asset (if a portfolio is consisted of *m* component assets). Thus, we have

$$f_j(x) = \sum_{i=1}^{k} \alpha_{ij} f_{ij} \tag{1}$$

where $\alpha_{ij} > 0$, $\sum_{i=1}^{k} a_{ij} = 1$, f_{ij} is a component Gaussian distribution $N(\mu_{ij}, \sigma_{ij}^2)$ in the Gaussian mixture distribution. We now consider the total return distribution during *m* periods. Therefore, generally speaking, the return distribution is a convolution distribution of all *m* periods, an m-fold convolution. For simplicity, and without loss of generality, we just show the case when *m=2*.

Consider the convolution theorem. That is, the convolution of two Gaussian distributions with different means and variances, say, $x_1 \sim N(\mu_1, {\sigma_2}^2)$ and $x_2 \sim N(\mu_2, {\sigma_2}^2)$, results in a new Gaussian distribution with mean and variance equal to the summations of each mean and variance, respectively, namely:

$$g(z) = \int_{-\infty}^{\infty} g_1(z-x) g_2(x) dx = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 + \sigma_2^2}} e^{-\frac{(z-(\mu_1+\mu_2))^2}{2\sqrt{\sigma_1^2+\sigma_2^2}^2}} \tag{2}$$

where $g_1(x)$ and $g_2(x)$ are the probability density function (p.d.f.) of random variables x_1 and x_2 , respectively.

Now consider the total return distribution within two periods, namely m=2. The p.d.f. of the total return within two periods then can be written as $p(z = x_1 + x_2)$, we then have:

$$p(z) = \int_{-\infty}^{+\infty} f_1(z - x) f_2(x) dx \tag{3}$$

However, the return distributions $f_1(x)$ and $f_2(x)$ are denoted as Gaussian mixture distributions, assuming the number of component Gaussian distributions are two, namely, k=2, we then have

$$x_1 \sim f_1(x) = \alpha_{11} f_{11} + \alpha_{21} f_{21}$$
$$x_2 \sim f_2(x) = \alpha_{12} f_{12} + \alpha_{22} f_{22}$$

It yields,

$$\begin{aligned} f(z) &= \int_{-\infty}^{\infty} f_1(z-x) f_2(x) dx \\ &= \int_{-\infty}^{\infty} [\alpha_{11} f_{11}(z-x) + \alpha_{21} f_{21}(z-x)][\alpha_{12} f_{12}(x) + \alpha_{22} f_{22}(x)] dx \\ &= \int_{-\infty}^{\infty} \alpha_{11}\alpha_{12} f_{11}(z-x) f_{12}(x) dx + \int_{-\infty}^{\infty} \alpha_{11}\alpha_{22} f_{11}(z-x) f_{22}(x) dx \\ &\quad + \int_{-\infty}^{\infty} \alpha_{21}\alpha_{12} f_{21}(z-x) f_{12}(x) dx + \int_{-\infty}^{\infty} \alpha_{21}\alpha_{22} f_{21}(z-x) f_{22}(x) dx \end{aligned} \tag{4}$$

Equation (4) can be expressed by the summation of the following four terms:

Term 1: $\alpha_{11}\alpha_{12} \int_{-\infty}^{\infty} f_{11}(z-x) f_{12}(x) dx$

which means it follows the following Gaussian distribution with the weight $\alpha_{11}\alpha_{12}$

$$\alpha_{11}\alpha_{12} N(\mu_{11} + \mu_{12}, \sqrt{(\sigma_{11}^2 + \sigma_{12}^2})^2) \tag{5a}$$

Term 2: $\alpha_{11}\alpha_{22} \int_{-\infty}^{\infty} f_{11}(z-x) f_{22}(x) dx$
which means it follows the following Gaussian distribution with the weight $\alpha_{11}\alpha_{22}$

$$\alpha_{11}\alpha_{22}N(\mu_{11}+\mu_{22},\sqrt{(\sigma_{11}^2+\sigma_{22}^2)^2}) \tag{5b}$$

Term 3: $\alpha_{21}\alpha_{12}\int_{-\infty}^{\infty} f_{21}(z-x)f_{12}(x)dx$

which means it follows the following Gaussian distribution with the weight $\alpha_{21}\alpha_{12}$

$$\alpha_{21}\alpha_{12}N(\mu_{21}+\mu_{12},\sqrt{(\sigma_{21}^2+\sigma_{12}^2)^2}) \tag{5c}$$

Term 4: $\alpha_{21}\alpha_{22}\int_{-\infty}^{\infty} f_{21}(z-x)f_{22}(x)dx$

which means it follows the following Gaussian distribution with the weight $\alpha_{21}\alpha_{22}$

$$\alpha_{21}\alpha_{22}N(\mu_{21}+\mu_{22},\sqrt{(\sigma_{21}^2+\sigma_{22}^2)^2}) \tag{5d}$$

Each term above indicates that each part of the convolution operation corresponds to a newly weighted normal distribution. Moreover, for the summation of each weight of each distribution, we have

$$\begin{aligned}&\sum w_{ij}=\alpha_{11}\alpha_{12}+\alpha_{11}\alpha_{22}+\alpha_{21}\alpha_{12}+\alpha_{21}\alpha_{22}\\&=\alpha_{11}(\alpha_{12}+\alpha_{22})+\alpha_{21}(\alpha_{12}+\alpha_{22})=(\alpha_{11}+\alpha_{21})(\alpha_{12}+\alpha_{22})=1\end{aligned} \tag{6}$$

This means that the convolution of Gaussian mixture distributions yields the same class of Gaussian mixtures as well, though, with some new distributional parameters. This result is also true in an m-fold convolution case. It is easier for one to use the mathematical induction to prove this general result.

Because we have obtained the convolution distribution of the total two periods, we now consider the probability of the return. The probability for the return of the total *m* periods larger than R can be estimated as

$$P(\sum_{j=1}^{m}x_j > R\ )=1-\int_{-\infty}^{R_0} f\ (z)dz \tag{7}$$

where *f(z)* is m-fold convolution density function, the return distribution of total m periods. Thus, the VaR can be also evaluated as

$$P(\sum_{j=1}^{m}x_j \le Q)=\int_{-\infty}^{Q} f(z)dz=\alpha \tag{8}$$

For example, for m=2, we have

$$P(x_1+x_2\le Q)=5\%$$

or inversely,

$$Q = F^{-1}(\alpha) = \inf\{x \in R \mid F(x) \geq \alpha) \tag{9}$$

Because the convolution distribution remains the same class of Gaussian mixture distributions, in Monte Carlo simulations one may simply generate the random numbers proportional to each component weight in the convolution distribution, to identify the return characteristics of a portfolio, or evaluate the VaR.

Furthermore, compared to other complicated distributions, in the case when one is applied to the return distribution of an asset, the Student distribution, for example. The convolution distribution of such a portfolio then turns out to be a complicated functional form. Usually a simple and distinct distribution function cannot be obtained under such a setting. Thus, it is difficult not only for one to do further theoretical research, but also in simulation studies, since it is not easy to generate random numbers from such a complicated convolution distribution function.

APPLICATION: RETURN AND THE VaR OF A PORTFOLIO OF M ASSETS

Suppose we have a portfolio of m assets and consider the return of the portfolio. We denote the investing weight on the jth asset as β_j, where $\sum_j \beta_j = 1$.

Let $z = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_m x_m$,

then we have

$$p(z) = p(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_m x_m) \tag{10}$$

By denoting $\beta_j x_j$ as z_j, we have the p.d.f. of z,

$$p(z) = p(z_1 + z_2 + \cdots + z_m) \tag{11}$$

It is an m-fold convolution as well. Since

$$\begin{aligned} x_1 &\sim f_1(x) = \alpha_{11} f_{11} + \alpha_{21} f_{21} + \cdots + \alpha_{k1} f_{k1} \\ x_2 &\sim f_2(x) = \alpha_{12} f_{12} + \alpha_{22} f_{22} + \cdots + \alpha_{k2} f_{k2} \\ &\cdots \\ x_m &\sim f_m(x) = \alpha_{1m} f_{1m} + \alpha_{2m} f_{2m} + \cdots + \alpha_{km} f_{km} \end{aligned} \tag{12}$$

For $z_j = \beta_j x_j$, it corresponds to a Gaussian mixture distribution as follows,

$$z_j \sim \alpha_{1j} N(\beta_j \mu_{1j}, \beta_j \sigma_{1j}^2) + \alpha_{2j} N(\beta_j \mu_{2j}, \beta_j \sigma_{2j}^2) + \cdots + \alpha_{kj} N(\beta_j \mu_{kj}, \beta_j \sigma_{kj}^2) \tag{13}$$

It yields $p(z) = p(z_1 + \cdots z_m)$ as a m-fold convolution with each Gaussian mixture distribution of z_j (j=1, 2,…, m).

Then, again, we get a new Gaussian mixture distribution from this convolution operation in just the manner noted above, and this new Gaussian mixture distribution remains the same Gaussian mixture class. Consequently, the probability for the return of this portfolio larger than R can be calculated as

$$P(\sum_{j=1}^{m} z_j > R\ ) = 1 - \int_{-\infty}^{R_0} f(z)dz \tag{14}$$

where *f(z)* is the m-fold convolution density function.

The VaR can be estimated as

$$P(\sum_{j=1}^{m} x_j \le Q) = \int_{-\infty}^{Q} f(z)dz = \alpha \tag{15}$$

$$Q = F^{-1}(\alpha) = \inf\{x \in R \mid F(x) \ge \alpha) \tag{16}$$

In our previous studies (Tan, 2005, Tan and Tokinaga, 2006, 2007), we have applied the Genetic Algorithm (GA) to optimize the weights of components Gaussian distributions. GA is known as a tool of Artificial Intelligence (AI), and it has the capability to find out the global optimal solution, not getting stuck in a local optimal solution. Besides, GA has been also applied to many fields from engineering to social issues. Other methods such as the Markov Chain Monte Carlo (MCMC) method can also be applied. One may choose one of the available methods and apply it to one's problem at hand.

CONCLUDING REMARKS

In this paper, we proposed a class of Gaussian mixture distributions to estimate the return distribution of a portfolio and applied it in risk management, estimation of Value at Risk (VaR) for a portfolio. In our previous works, we have shown that a complicated return distribution having non-normal characteristics, such as heavy-tailed behavior, skewed distributional shape etc, can be accurately approximated using a class of Gaussian distributions (Tan and Tokinaga 2007a). Meanwhile various numerical applications, even for multimodal cases, have confirmed the effectiveness and accuracy of our proposed method (Tan and Tokinaga, 2007b, Tan, et al., 2011).

In this work, we extend our previous results and apply our method to the return of a portfolio and VaR. We have theoretically shown that a convolution distribution (return distribution of a portfolio) of several Gaussian mixture distributions (return distributions of component assets) yields a Gaussian mixture distribution as well. Such a good statistical property makes the model easily tackled. For example, in simulation studies, one may generate random numbers easily from such a class of Gaussian mixtures, by simply generating the random numbers proportional to the weight of each component Gaussian distribution, even in the case of estimating some rare events, such as the Value at Risk. Such a class of Gaussian mixtures can represent those non-normal phenomena in the return distribution of a portfolio, such as heavy-tailed behavior, skewness, and excess kurtosis accurately and keep the model simple.

It is no longer necessary to introduce any complicated distribution family, such as, the Student t, Generalized Error Distribution, to capture the statistical characteristics of financial returns, since it is difficult to fit the parameters in a complicated distribution and hard to accurately catch the statistical characteristics of returns using a single distribution family. Meanwhile, the use of a complicated distribution could introduce a convolution distribution for a portfolio with serious complicated function form, and makes it difficult to utilize in both academic research and business practice. However, using our proposed method, once the Gaussian mixture distribution of each individual asset is identified, one can obtain the return distribution of a portfolio. Namely, the combinational weights, and the distributional parameters (mean and variance of each component Gaussian distribution) are obtained automatically by our proposed convolution approach.

We have stated that a Gaussian mixture distribution can be estimated by Genetic Algorithm, or the Markov Chain Monte Carlo simulation. However, with the increased number of components the convergence speed for the parameters estimation could be greatly decreased. Future research will design and develop some parallel computation algorithms to solve this time-consuming problem.

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