VALUE AT RISK ESTIMATION FOR HEAVY TAILED DISTRIBUTIONS

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ABSTRACT

The aim of this paper is to derive a coherent risk measure for heavy tailed GARCH processes using extreme value theory. For the proposed measure, the risk associated to a given portfolio is less than the sum of the stand-alone risks of its components. This measure which is value at risk (VaR), is the limiting result of an infinity shift of location and is less sensitive with respect to location change. Based on two international stock markets applications and an empirical backtesting procedure, the proposed VaR is found to be more accurate in all quantile levels.

JEL: C22, C58, G15

KEYWORDS: Risk Management, Extreme Value Theory, Non-linear Models, Backtesting, Stock Market Index

INTRODUCTION

SEveral authors argue Value at Risk (VaR) is the best integrated risk management tool in both financial and insurance studies. VaR is defined as an amount lost in a portfolio with a given small probability over a fixed number of days. Besides its simplicity and intuitive interpretation, VaR works across different asset classes such as stocks and bonds. However, this tool has numerous shortcomings. First, for several high frequency time series data characterized by thick-tail, such as stock returns, if the data are still supposed following normal distribution, the VaR would be underestimated. Second, the normal distribution hypothesis is always rejected by Jarque-Bera test in empirical analysis. Therefore, how to deal with these two limitations is paramount in risk management. In order to avoid these two first shortcomings, one possible solution is based on non-parametric methods that make no assumptions concerning the nature of the empirical distribution function. The third VaR drawback is the stylized fact namely heterocedasticity phenomena that describe financial data. The latter often exhibit volatility clustering or persistence. In volatility clustering, large changes tend to follow large changes, and small changes tend to follow small changes. To explain these features of the data, Bollerslev (1986) proposed the popular GARCH models. The latter takes into account volatility clustering and excess kurtosis (fat tail behavior) which are considered as the forth VaR shortcoming. To overcome the problem of fat tail behavior, one can use extreme value theory (EVT). This theory is an interesting tool to deal with extreme observations in order to measure the density in the tail. Combining with GARCH model, EVT has the different statistical characteristics to describe the performance of the tick-tail properties of the high frequency financial time series data.

The remainder sections are organized as follows. Section 2 outlines VaR concept based on EVT as proposed by Dekkers et al.(1989). The proposed modified VaR is introduced in Section 3. Backtesting procedure is outlined in section 4. In Section 5, the proposed method is illustrated through a real case study and backtesting methodologies. Finally, conclusion is provided in section 6.

LITERATURE REVIEW

There is a growing literature on application of EVT approaches to estimate VaR. In series of articles, McNeil (1997, 1998, and 1999) proposed to use the tail index in order to estimate VaR for the financial time series using EVT. Silva and Mendes (2003) show that VaR estimation based on EVT is more conservative to determine capital requirements than traditional methods. Using daily returns, Gencay et al. (2003) indicate that GARCH and generalized Pareto distribution (GPD) models are more preferable than several others traditional models for most quantile levels. Maghyereh and Al-Zoubi (2006) investigate performances of some models to estimate VaR in seven Middle East and North Africa (MENA) countries. They outline that EVT models perform better in five of the MENA stock markets. Alper et al. (2007) compare the performance of eight filtered EVT models with those of GARCH and FIGARCH models. Based on backtesting, they outlined that EVT models perform better than the competing parametric models. Zikovic and Aktan (2009) investigate the relative performance of some VaR models with the daily returns of Turkish and Croatian stock. They indicate that during the crisis period, all tested VaR models, except EVT and hybrid historical simulation models, seriously under-predict the true level of risk. Ouyang (2009) calculates the VaRs of daily returns of Shanghai and Shenzhen indexes using equally weighted moving average, GARCH(1,1) empirical density estimation method and GPD models. The method based on EVT, produces in all cases considerably higher VaR estimates than any other approaches. To have further enquiry about the application of GARCH and/or EVT approaches to estimate the VaR, one can see Bali (2003), Huang and Lin (2004), Jones et al. (2004), Zhao et al. (2009), Neftci (2000) and Pesaran and Zaffaroni (2004).

The validation of any proposed VaR is a significant issue in the acceptance of VaR models for market risk management. The tests for validating models used here are the backtesting procedure proposed by Basel Committee (1996) .For example, the basic frequency of tail losses (or Kupiec) test and the conditional Christoffersen (1998) approach, which is based not only on the frequency of observed exceptions but also on the independence among them.

Following the above references, understanding the influence of extreme market events in volatility periods is of great importance for risk managers. Thereof, we propose an estimation of VaR based on EVT-GARCH combination. The proposed measure is based on the mathematical formula given by Dekkers et al.(1989) and modified by Vermaat et al. (2005) and it take into consideration the stylized facts of financial data especially the volatility clustering and the leverage effect. The proposed VaR has an interesting location property of invariance and take into consideration the stylized facts of financial series.

Theoretical Development

In this section, we focus on the maximal relative return losses. Extreme can be defined as the maximum of random variables $Z_1,...,Z_T$: $M_T = \max_{t=1,...,T} Z(t)$, of which fluctuations may be characterized by a generalized extreme value (GEV) distribution. We use this distribution to approximate the distribution of suitably normalized extreme return losses $\frac{M_T - \mu}{\sigma}$ where σ and μ are, respectively, the scale and location parameters. The GEV distribution function is given b

$$F_\gamma(x) = \exp\left\{-(1+\gamma x)^{-1/\gamma}\right\},\ \ 1+\gamma x \geq 0,\ \gamma \in \Re \tag{1}$$

which is interpreted as $\exp\{e^{-x}\}$ for $\gamma = 0$. The question is how to estimate γ from a finite sample $Z_1,...,Z_T$ drawn from its unknown distribution function *F* regarding to its maximum value. A well-known result in EVT is that there are only three types of possible limit distributions for the maximum of i.i.d. random variables under positive affine transformations, depending on the tail behavior of their common density. The limit distribution of the normalized maxima, when $n \to \infty$, was then proved by Fischer and Tippett (1928) and Gnedenko (1943) to be either the Gumbel, the Weibull, or the Fréchet distribution which are respectively related to $\gamma = 0$, to $\gamma < 0$, and to $\gamma > 0$. Let $Z_{(1)} < Z_{(2)} < .... < Z_{(T)}$ denote the order statistics of the returns sequence $Z_1,...,Z_T$ and *m* the number of the largest order statistics. When $\gamma > 0$, one can use Hill (1975)'s estimate defined as

$$M_T^{(1)} = \frac{1}{m}\sum_{j=1}^{m} \ln Z_{(T-j+1)} - \ln Z_{(T-m)} \tag{2}$$

Hill estimator is proven in Mason (1982) to be a consistent estimator of γ for fat-tailed distributions. When no assumption being made on the tail index γ, one may use the moment estimator of Dekkers et al. (1989) for quantiles of large order defined by

$$\hat{\gamma}_T = M_T^{(1)} + 1 - \frac{1}{2}\left\{1 - \frac{(M_T^{(1)})^2}{M_T^{(2)}}\right\}^{-1} \tag{3}$$

where

$$M_T^{(2)} = \frac{1}{m}\sum_{j=1}^{m} (\ln Z_{(T-j+1)} - \ln Z_{(T-m)})^2 \tag{4}$$

The Hill estimator and the moment estimator of Dekkers et al. (1989) have the best performance especially for the case $\gamma > 0$ (see Tsourti and Panaretos (2001) and Pictet et al. (1996)). However, even these estimators are scale invariant, they are not location invariant. The $(1-\alpha)$ quantile of the distribution function F_γ for $0 < \alpha < \frac{1}{2}$ is given by

$$F_\gamma(1-\alpha) = X_{(T-m)} + \frac{({}^{m}/_{T_\alpha})^{\hat{\gamma}_T}}{\hat{\gamma}_T}(1 - \min(\hat{\gamma}_T - 0))X_{(T-m)}M_T^{(1)} = VaR_{\text{Dekkers}} \tag{5}$$

For industrial production processes, Vermaat et al. (2005) proposed another approach to compute quantiles for largest and smallest order statistics which are a limiting result of infinity shift of location of the estimators proposed by Dekkers et al. (1989). For largest order statistic, the $(1-\alpha)$ quantile is:

$$F_T^{-1} = Z_{(T-m)} + D(\frac{m}{Tq})\frac{1}{m}\sum_{j=1}^{m}(Z_{(T-j+1)} - Z_{(T-m)}) = VaR_{Vermaat}, \tag{6}$$

with,

$$D = \frac{(\frac{m}{Tq})^G - 1}{G}(1 - \min(G,0)),$$

$$G = 1 - \frac{1}{2(1-Q)},$$

$$Q = \frac{\left[\frac{1}{m} \sum_{j=1}^{m} \left\{ Z_{(T-j+1)} - Z_{(T-m)} \right\} \right]^2}{\frac{1}{m} \sum_{j=1}^{m} \left\{ Z_{(T-j+1)} - Z_{(T-m)} \right\}^2}$$

The advantage of estimators proposed by Vermaat et al. (2005) is that the corresponding estimates of large and small order are symmetric around the mean for a symmetric distribution.

As outlined above, the VaR proposed by Dekkers et al.(1989) and modified by Vermaat et al. (2005) did not take into consideration the stylized facts of financial data especially the volatility clustering and the leverage effect. To deal with this problem, the following section proposes to use the EVT/GJR-GARCH combination to calculate a modified VaR where a shift of location is introduced.

DATA AND METHODOLOGY

It is important to note that the VaRs proposed by Dekkers et al.(1989) and Vermaat et al.(2005) depend on an assumption of i.i.d. observations. Clearly, this is not true for financial time series. To deal with this problem, we propose to use McNeil and Frey (2000)'s approach. Based on two stages, we estimate a GARCH model to original data by quasi-maximum likelihood in stage one and filter the original series to obtain i.i.d. residuals. In stage two, we consider the residuals computed in Step 1, and estimate the tails of the innovations using EVT. Then, calculate the modified VaR. The advantage of this GARCH-EVT combination lies in its ability to capture conditional heteroscedasticity in the data through the GARCH framework, while at the same time modeling the extreme tail behavior through the EVT method. The use of GARCH models proposed is justified by its usefulness in detecting stylized facts. A GARCH (p,q) model is defined as follow,

$$X_t = \sigma_t Z_t \tag{7}$$

with

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i . X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 , \tag{8}$$

and $\alpha_0 > 0$, $\alpha_i > 0$, $i = 1,..., p$, $\beta_j \geq 0$, $j = 1,..., q$ and Z_t are i.i.d. innovations.

This model can be written by successive iteration,

$$\sigma_t^2 = \omega \left\{ 1 + \sum_{T=1}^{\infty} \prod_{i=1}^{T} \alpha_1 Z_{t-i}^2 + \beta_1) \right\}, \tag{9}$$

In the presence of shifts, the GARCH (1,1) model is given as follow

$$X_t^* = \sigma_t^* Z_t^* \tag{10}$$

where,

$$Z_t^* = L_T^* (Z_T + K) \tag{11}$$

$$\sigma_t^{*2} = \omega + \alpha_1 X_{t-i}^{*2} + \beta_1 \sigma_{t-1}^{*2} , \tag{12}$$

where *K* is the magnitude of the shift.

Our methodology is as follow. First, we use the tail index estimator to compute the VaR estimation. Second, we shift the innovations with a constant *K*. For the transformed observations, we estimate the VaR with Dekkers et al.(1989)'s method.

Results are given in the following Proposition (for $K \to \infty$).

Proposition

Define $Z_t^* = Z_t + K$, where Z_t are i.i.d. random variables, and *K* is a magnitude of the shift. Suppose that $\frac{m}{T(1-\alpha)} \geq 1$. If $K \to \infty$, then the proposed modified VaR with $0 < \alpha < \frac{1}{2}$ is

$$VaR_\alpha = \left\{ \frac{\omega}{1-\alpha_1-\beta_1} \right\}^{0.5} \left\{ Z_{(T-m)} + D(\frac{m}{T(1-\alpha)}) C_m(T) \right\}, \tag{13}$$

for given ω, with,

$$D(\frac{m}{T(1-\alpha)}) C_m(T) = \frac{\left\{ \frac{m}{T(1-\alpha)} \right\}^{G_T} - 1}{G_T} \{1 - \min(0, G_T)\},$$

$$G_T = 1 - \frac{1}{2(1-Q_T)}$$

$$C_j = Z_{(T-j+1)} - Z_{(T-m)},$$

$$C_m(T) = \frac{1}{m} \sum_{j=1}^{m} \{ Z_{(T-j+1)} - Z_{(T-m)} \},$$

and

$$Q_T = \frac{\left[\frac{1}{m} \sum_{j=1}^{m} C_j \right]^2}{\frac{1}{m} \sum_{j=1}^{m} (C_j)^2}$$

The proof of the proposition is given in the Appendix.

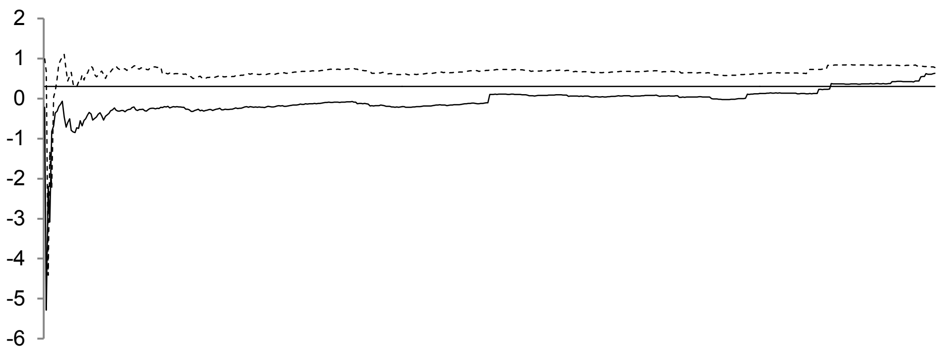
Following the above proposition, the VaR computation is a function of GARCH model parameters, quantile order and the number of exceedances (*m*) beyond a threshold μ. This Proposition can be extended to take into account all models of GHARCH family as GJR-GHARCH and QGHARCH.

The tail index estimator defined in $G_T = 1 - \frac{1}{2(1-Q_T)}$ will be refereed as the modified moment estimator since it stands for the limiting value of the moment estimator as the shift goes to infinity. From our experience with several data sets, as a rule of thumb, the proposed VaR provides a rather better approximation of the actual tail index than the classical moment estimator. As shown by Fig. 1, the modified tail index estimator converges to the true value of the shape parameter of a sample of size T = 5000 generated from a GEV distribution with parameters $\mu = 0.5$, $\sigma = 0.1$ and $\gamma = 0.3$; while the original moment estimator is significantly above the actual value, thus overestimating $\gamma = 0.3$.

Backtesting

The validation of risk measures is a significant issue in the acceptance of VaR models for market risk management. Basel Committee on Banking Supervision (1996) specifies that "the essence of all backtesting efforts is the comparison of actual trading results with model-generated risk measures". The tests for validating models used here are those validate if the number of observed exceptions is consistent with the number of expected exceptions. This type of tests are proposed by Basel Committee's procedure, the basic frequency of tail losses (or Kupiec) test and the conditional Christoffersen (1998) approach, which is based not only on the frequency of observed exceptions but also on the independence among them.
Kupiec (1995) focused on the property of unconditional coverage. This kind of backtests are concerned with whether or not the reported VaR is violated more than α of the time. In cases where the VaR has been underestimated and thus when the process has experienced a value greater than VaR, we say that VaR has been violated, and such an event is called a violation of VaR.

Figure 1: Tail Index Estimates From Simulated Data: Moment Estimator (dashed line) and Modified Moment-Estimator (solid line)



If there have been too many violations, then the VaR model may not be adequate for the instruments composing the sample. If we observe a time series of past ex-ante VaR forecasts and past ex-post X_t, a hit sequence function cause defined as $I_{t+1} = 1$ if $X_t < VaR_t^{\alpha}$ and $I_{t+1} = 0$ if $X_t > VaR_t^{\alpha}$. Then, we have a sequence $\{I_t\}_{t=1}^{T}$ across T times indicating when the past violations occurred. If we could predict the VaR violations, then that information could be used to construct a better model. The hit sequence of violations is distributed as Bernouilli variable. In order to test if *p*, the fraction of violations obtained is significantly different from the promised fraction α. Based on unconditional coverage hypothesis and under H_0. The likelihood of an i.i.d. Bernouilli (*p*) hit sequence is

$$L(\pi) = \prod_{t=1}^{T} (1-\pi)^{1-I_{t+1}} \pi^{1-I_{t+1}} = (1-\pi)^{T_0} \pi^{T_1} = (1-\alpha)^{T_0} \alpha^{T_1}, \quad (14)$$

where T_0 is the number of 0 and T_1 is the number of 1 in the sample. Let $\hat{\pi} = \frac{T_1}{T}$, the observed fraction of violation in the sequence.

The likelihood ratio test is

$$LR = -2\ln\left\{\frac{L(\alpha)}{L}\right\} = -2\ln\left\{\frac{(1-\alpha)^{T_0}\alpha^{T_1}}{(1-\frac{T_1}{T})T_0(\frac{T_1}{T})^{T_1}}\right\} \rightarrow \chi^2 \quad (15)$$

If the p-value is below the desired significance level, we will reject the null hypothesis.

The alternative approach developed by Christoffersen (1998) is to estimate a confidence interval to the number of exceptions based on the available sample and verify whether the observed number of exceptions is consistent with the forecasted, including an independence test. He suggests a procedure to evaluate the precision of predictions in confidence intervals, which tries to capture the VaR estimative conditionality. This test is

$$LR = -2\ln\left\{\frac{L(\alpha)}{L(\hat{\pi})}\right\} = -2\ln\left\{\frac{(1-\alpha)^{T_0}\alpha^{T_1}}{(1-\pi_{01})^{T_0}(\pi_{01})^{T_{01}}(1-\pi_{11})^{T_{01}}(\pi_{11})^{T_{11}}}\right\} \rightarrow \chi^2, \quad (16)$$

where T_{10} is the number of times a violation is followed by a good return ; T_{11} is the number of times a violation is followed by another violation; T_{01} is the number of times a good return is followed by a violation; and T_{00} is the number of times a good return is followed by another good return. Also let

$$\pi_{01} = \frac{T_{01}}{T_{00}+T_{01}} \quad (17)$$

and

$$\pi_{11} = \frac{T_{11}}{T_{10}+T_{11}} \quad (18)$$

Results will be presented in the real case study.

RESULTS AND DISCUSSION

The empirical analysis is based on two international stock market indexes, CAC40 and DAX. We use daily return series from March 15- 1991 to January 21- 2009. We end up with balanced data set of 4500 daily returns for two stock markets which makes 9000 logarithmic returns measured in percentage terms and denoted as:

$$r_t = 100(\ln P_t - \ln P_{t-i}), \quad (19)$$

with P_t the original prices at time t.

Table 1 present's summary descriptive statistic of CAC40 and DAX indexes and shows considerable similarities between the two indexes. But, the DAX index indicates that Germany stock market experienced the most severe market pressure and turbulence.

The skewness and kurtosis indicate that the two stock market index distributions have fat tails. CAC40 and DAX indexes are skewed to the right with kurtosis coefficient exceeding the value of three found for the normal distributions. Jarque Bera statistics give overwhelming evidence that the normality null hypothesis is rejected in all cases.

These results a priori indicate that classical VaR models based on normal distribution have a hard time forecasting the true level of risk. Given these characteristics, EVT-VaR models should be more performant for capturing the true level of risk since they focus on the tail regions of the return distributions.

Table 1: Summary Descriptive Statistics For CAC40 and DAX For the Period January 2003- November 2009

Descriptives Statistics	**CAC40**	**DAX**
Mean	0.0246	0.0321
Maximum	8.225	7.552
Minimum	-10.137	-13.709
Std. Dev.	1.295	1.393
Skewness	0.1524	0.3325
Kurtosis	7.557	9.091
Jarque Bera	5,039.1	99,472
Probability	0.0000***	0.0000***

This table shows summary descriptive statistic of CAC40 and DAX indexes.

Table 1 is done in order to show the distributions characteristics, precisely, show the index behavior which is different to the normal case. It can be readily seen that the two investigated indexes have shown a distribution behavior adequate to the hetroscedastic model. The values in the Table shows that the European stock markets have the same characteristics for the period that span from 1991 to 2009 that includes crisis and stable periods.

A preliminary step in proceeding with EVT analysis is to examine the unit root property of the two indexes. Table 2 shows the results of the Augmented Dickey-Fuller (ADF) which tests the null hypothesis that the stock indexes has unit root against the stationarity alternative in the two stock market indexes. The ADF test confirms that the logarithmic returns of CAC40 and DAX are two stationary variables.

Table 2: Augmented Dickey-Fuller Unit Root Test

Sig. level	**p-value (CAC40)**	**p-value (DAX)**
0.0100	0.0000***	0.0000***
0.0500	0.0010***	0.0020***
0.1000	0.0010***	0.0050***

*This Table summarizes the Augmented Dickey-Fuller Unit Root Test for the CAC40 and DAX daily returns. The test is done for three different significance level, it shows that the null hypothesis is rejected in all cases. *** indicate significance at 1%*

An AR(1)-GARCH(1,1) model is filtered for each stock market index in order to remove autocorrelation and to capture the conditional heteroskedasticity by incorporating asymmetric leverage effects for volatility clustering. The reason of using non linear autoregressive model for the two indexes is explained a detailed analysis of the residuals series. The residual are iid (0, 1) process upon which EVT estimation of the sample cumulative function (CDF) tails is based. The non-linear model is supported from Q-test where squared returns are used. This test confirms that there is autocorrelation in the second moment. Additionally, we run Engle's ARCH-test which confirms the presence of ARCH effects in our time series. We fitted the asymmetric model as:

$$r_t = c + \phi_i r_{t-1} + \varepsilon_t \tag{20}$$

$$\varepsilon_t = \sigma_t Z_t$$

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \psi_1 I_{\{\varepsilon t-1 \le 0\}} \varepsilon_{t-i}^2 + \beta_1 \sigma_{t-1}^2, \tag{21}$$

Where:

$$\varepsilon_t = r_t - \mu, \; Z_t \text{ is iid } (0, 1) \tag{22}$$

The model given by (12) data set starting with low orders of GARCH and then selected the ones that fitted the data better. We tried models of different order for each time series and among the ones with significant coefficient we made the selection based on likelihood ratio (LR) test. One competing model is chosen: AR (1)- GARCH (1, 1) for CAC40 and DAX data. The LR test and the final parameters for each time series are summarized in Table 3.

Table 3 shows the estimated parameters of the specified AR (1)- GARCH (1,1) model. It is clear that the stationarity conditions still verified for both indexes. Table 3 summarizes the estimated parameters of the fitted AR(1) GARCH (1,1) to the CAC40 and DAX indexes. It is proved that all the estimated parameters are significant at 1%. In addition, it is clear that the stationary conditions still verified.

Table 3: AR (1) - GARCH Models Estimation

	Coefficient	**Std.Error**	**p-value**
CAC40			
c	0.0353***	0.0068	0.0000
ω	0.0020***	0.0005	0.0000
ϕ_1	-0.0003***	0.0157	0.0000
α_1	0.1274***	0.0086	0.0000
β_1	0.9328***	0.0054	0.0000
DAX			
c	0.0413***	0.0063	0.0000
ω	0.0027***	0.0004	0.0000
ϕ_1	0.0064***	0.0161	0.0000
α_1	0.1571***	0.0097	0.0000
β_1	0.9126***	0.0059	0.0000

*This table shows the estimated parameters of the specified AR (1)- GARCH (1,1) model. *** indicate significance at 1%*

For VaR accuracy test, unconditional coverage tests and conditional coverage tests were performed for both Dekkers's and our proposed models. Results are summarized in Tables 4 and 5. For the proposed VaR, Table 4 shows that the two stock market indexes have a p-value below each value of α . Therefore, we can reject the null hypothesis at 1%, 0.5% and 0.1% levels. However, for the Dekkers's VaR, the null hypothesis cannot be rejected (see Table 5). Then, Backtesting ensured that the proposed VaR achieved excellent results.

Table 4 shows the VaR performance evaluation, for this aim the kupiec and christofersen tests are elaborated, it shows that the proposed VaR is accurate except for, 0.5% and 0.1% for both indexes. Through this table, we can conclude that the Backtesting results based on Kupiec test and Christofersen test for the two stock indexes, confirms that the proposed VaR is accurate in different confidence levels. For instance, at the confidence level, 98% the Kupiec test value is about (0.002) and (0.003) for the

CAC40 and DAX respectively, which is near to the risk level that confirm the accuracy of the proposed VaR.

Table 4: Backtesting of the Proposed VaR

Index Daily Returns	Confidence Level	Kupiec Test (p-value)	Christoffersen Test (p-value)
CAC40	**99.000**	**0.0000*** **	**0.0000*** **
	98.000	0.0020***	0.0100**
	97.000	0.0000***	0.0000***
	96.000	0.0130**	0.0110**
	95.000	0.0000***	0.0000***
DAX	**99.000**	**0.0010*** **	**0.0000*** **
	98.000	0.0030***	0.0140**
	97.00	0.0040***	0.0110**
	96.00	0.0230**	0.0220**
	95.00	0.0120**	0.0240**

*This table shows the VaR performance evaluation *** indicate significance at 1%, ** indicate significance at 5%*

Table 5 shows the performance evaluation of the Dekkers's VaR, the kupiec and Christofersen tests shows that the VaR is accurate at different significance levels for both indexes. For instance, the Kupiec test value in 99% level (0.140) and (0.1412) for the CAC40 and DAX indexes respectively shows the performance of the Dekker's VaR model. This result is also confirmed referring to the Christoferssen test values (0.1111) and (0.1241) respectively.

Table 5: Backtesting of Dekkers's VaR

Index Daily Returns	Confidence	Kupiec Test (p-value)	Christofferse N Test (p-value)
CAC40	99%	0.1400	0.1111
	98%	0.1330	0.1292
	97%	0.0611*	0.0731*
	96%	0.0182**	0.0160**
	95%	0.0000***	0.0000***
DAX	99%	0.1412	0.1241
	98%	0.1333	0.1342
	97%	0.1144	0.1111
	96%	0.0390**	0.0232**
	95%	0.0290**	0.0210**

*This table shows the performance evaluation of the Dekkers's VaR, the kupiec and Christofersen tests *** indicate significance at 1% ** indicate significance at 5%, ** indicate significance at 5%*

For implementation of our approach, estimation of the tail indexes parameters is crucial. Table 6 gives, for each market index, the number of order statistic (m), the Dekkers tail index $\hat{\gamma}_T$ and the modified Dekkers tail $\hat{G}_T$ index.

Table 6 shows the estimated Deckkers and the modified Dekkers tail index parameter for the CAC40 and DAX indexes. The parameter m is about (240) for the CAC40 series and (318) for the DAX series, it represents the number of the largest order statistics. The modified tail index estimator (0.1301) and (0.1202) for the CAC40 and DAX series respectively based on the proposed VaR provides a rather better approximation of the empirical tail index than the classical moment estimator (01800) for both series. Thus, the modified tail index estimator performs better than the other one and converges to the emperical value of the investigated sample.

Table 6: Extreme Value Statistics Based on Daily Returns from CAC40 and DAX Stock Markets

Index Daily Returns	m	$\hat{\gamma}_T$	$\hat{G}_T$
CAC40	240.00***	0.1800***	0.1301***
DAX	318.00***	0.1800***	0.1202***

*This table shows estimated Deckkers and the modified Dekkers tail index parameters for the CAC40 and DAX indexes.*** *indicate significance at the 1% level.*

The proposed VaR estimation for the CAC40 and DAX returns is given in Table 7 and 8. Following these tables, it is clear that the proposed approach based on the combination of GARCH-EVT is more appropriate for calculating risk measures in all quantile levels. Then, Table (7) and (8) show that the proposed VaR is very close to the empirical one for different confidence levels between 0.95 and 0.99. This result indicates that the proposed method provides a better approximation of VaR than the standard VaR proposed by the literature.

Table 7 shows the comparison between the proposed VaR, the empirical VaR and the Dekkers VaR for different significance level for the CAC40 index, it is clear that the proposed VaR is the nearest one to the empirical VaR. It also shows that the proposed VaR is more accurate than the Dekker's VaR; this can be confirmed at all significance levels. For instance in the case of 95% referring to the empirical VaR,(1.962) the proposed VaR value is about (1.931) however the Dekker's VaR is about (2.531) thereby, it is clear that the proposed VaR is performant. In addition the same conclusion can be conducted in the 99% confidence level where, the proposed VaR value is about (3.543) that approximate the empirical VaR however, the Dekker's VaR overestimate it.

Table 7: Dekkers's, Empirical and Proposed VaR Comparison: CAC40 Index

Confidence Level	Empirical VaR	Dekkers's VaR	Proposed VaR
95%	1.962	2.531	1.931
96%	2.109	2.721	2.135
97%	2.323	2.976	2.407
98%	2.640	3.359	2.807
99%	3.440	4.083	3.543

This table shows the comparison between the proposed VaR, the empirical VaR and the Dekkers VaR for different significance level for the CAC40 index.

Table 8 shows the comparison between the proposed VaR, the empirical VaR and the Dekkers VaR for different significance levels for the DAX index, from 95% to 99% it still verified that the proposed VaR is more accurate than the Dekkers VaR. From this table, we can confirm that the proposed VaR perform well comparing to the Dekker's VaR,that underestimate the benchmark at all significance levels.

Table 8: Dekkers's, Empirical and Proposed VaR Comparison: DAX Index

Confidence level	Empirical VaR	Dekkers's VaR	Proposed VaR
95%	2.072	0.8064	2.164
96%	2.207	0.7970	2.354
97%	2.414	0.7877	3.010
98%	2.869	0.7785	3.360
99%	3.650	0.7695	4.064

This table shows the comparison between the proposed VaR, the empirical VaR and the Dekkers VaR.

For instance in the case of 95% referring to the empirical VaR,(2.072) the proposed VaR value is about (2.164); however, the Dekker's VaR is about (0.8064), Moving to the 99% confidence level the proposed

VaR value is about (4.064) however, the Dekker's VaR underestimate the empirical one (3.650).thus, we can confirm that the proposed VaR is more accurate.

CONCLUDING COMMENTS

The high volatility of the international stock market indexes requires the implementation of effective risk management. EVT is a powerful tool to estimate the extreme events' effects in risky markets based on non linear background statistical methodology. This study exhibits how EVT in conjunction with GARCH processes can be used to model tail-related risk measures such as VaR by applying it to French and Germany stock market daily indexes. In this paper, a modified VaR estimation are proposed based on quantile estimators of Dekkers et al.(1989). The proposed VaR are a limiting result of an infinity shift of location and is less sensitive with respect to location change. From our experience with several data sets, as a rule of thumb, the proposed VaR provides a rather better approximation of the actual tail index than the classical moment estimator. As the backtesting results show that the proposed approach is more accurate in all quantile levels, a real case study confirm that, for CAC40 and DAX stock market indexes, the modified VaR outperforms the Dekkers's one. Therefore, the proposed VaR is very close to the empirical one and is resistant when a shift does change the properties of the estimator as the case of location change. The proposed work can be extended to take into account all models of GHARCH models family such as EGHARCH, QGHARCH and TGARCH models.

APPENDIX

Proof of Proposition

$$VaR_\alpha(X_t) = \sigma_{t+1} VaR_\alpha(Z_t),$$

By replacing Z_t with Z_t^* the definitions of $\hat{\gamma}_t$, $M_t^{(r)}$ and σ_{t+1} are modified in

$\hat{\gamma}_t^*$, $M_t^{(r)*}$ and σ_{t+1}^*.

where

$$Z_t^* = L_t^*(Z_t + K),$$
$$L_t^* = \frac{1}{\sum_{j=1}^{m} Z_{(t-m)} - Z_{(t-j+1)}} \sum_{j=1}^{m} \ln \frac{Z_{(t-j+1)} + K}{Z_{(t-m)} + K} \quad (23)$$

The (1- α)-quantile of the distribution function of ε^* can be easily obtained from

$$VaR_\alpha(X_t^*) = \sigma_{t+1}^* VaR_\alpha(Z_t^*),$$

The VaR is derived as the limiting value for $K \to +\infty$ of σ_{t+1}^* and $VaR_\alpha(Z_t^*)$ where

$$VaR_\alpha(Z_t^*) = Z_{(t-m)} + \frac{\left(\frac{m}{t(1-\alpha)}\right)^{\hat{\gamma}_t^*} - 1}{\hat{\gamma}_t^*}(1 - \min(0, \hat{\gamma}_t^*))(Z_{(t-m)} + K) M_t^{(1)*} \quad (24)$$

First we calculate the following limit

$$\lim_{K \to +\infty} \sigma_t^{2*} = \left\{ \frac{\omega}{1 - \alpha_1 - \beta_1} \right\}^{0.5} \quad (25)$$

Now we calculate the limit of $\hat{\gamma}_t^*$,

$$\lim_{K \to +\infty} \hat{\gamma}_t^* = \lim_{K \to +\infty} M_T^{(1)*} + 1 - \frac{1}{2} \left\{ 1 - \frac{(M_T^{(1)*})^2}{M_T^{(2)*}} \right\}^{-1}.$$

So, we have to consider

$$\frac{(M_T^{(1)*})^2}{M_T^{(2)*}} = \frac{\left\{ \frac{1}{m} \sum_{j=1}^{m} \ln \frac{Z^*_{(t-j+1)}}{Z^*_{(t-m)}} \right\}^2}{\frac{1}{m} \sum_{j=1}^{m} \left\{ \ln \frac{Z^*_{(t-j+1)}}{Z^*_{(t-m)}} \right\}^2}$$

We develop $\ln \frac{Z^*_{(t-j+1)}}{Z^*_{(t-m)}} = \ln \left(1 + \frac{Z_{(t-j+1)} - Z_{(t-m)}}{Z_{(t-m)} + K} \right) = \ln\left(1 + C_j H\right),$

where, $C_j = Z_{(t-j+1)} - Z_{(t-m)}$ *and* $H = \frac{1}{Z_{(t-m)} + K}$.

Since $K \to +\infty$ *we have* $H \to 0,$ *and hence* $\lim_{H \to 0} \frac{\ln\left(1 + C_j H\right)}{H} = C_j$. This implies that,

$$\lim_{H \to 0} \frac{\frac{1}{m} \sum_{j=1}^{m} \ln\left(1 + C_j H\right)}{H} = \frac{1}{m} \sum_{j=1}^{m} C_j = C_m(t)$$

and

$$\lim_{H \to 0} \frac{\frac{1}{m} \sum_{j=1}^{m} \left\{\ln\left(1 + C_j H\right)\right\}^2}{H^2} = \frac{1}{m} \sum_{j=1}^{m} C_j^{\,2}$$

Then, we find that,

$$\lim_{H\to 0}\frac{(M_T^{(1)*})^2}{M_T^{(2)*}}=\frac{\left\{\frac{1}{m}\sum_{j=1}^{m}C_j\right\}^2}{\frac{1}{m}\sum_{j=1}^{m}C_j^{\,2}}=Q_t \tag{26}$$

We conclude from (27),

$$\lim_{K\to+\infty}\hat{\gamma}_t^{\,*}=1-\frac{1}{2(1-Q_t)}=G_t \tag{27}$$

with $G_t\in(-\infty,\frac{1}{2})$ with probability one.

Now we calculate the following limits

$$\lim_{K\to+\infty}KM_t^{(1)*}=\lim_{K\to+\infty}\frac{KH}{m}\sum_{j=1}^{m}\frac{\ln(1+C_jH)}{C_jH}=C_m(t)$$

and

$$\lim_{K\to+\infty}\frac{\left(\frac{m}{t(1-\alpha)}\right)^{\hat{\gamma}_t^{\,*}}-1}{\hat{\gamma}_t^{\,*}}(1-\min(0,\hat{\gamma}_t^{\,*}))=D\left(\frac{m}{t(1-\alpha)}\right) \tag{28}$$

Where: $D\left(\frac{m}{t(1-\alpha)}\right)=\frac{\left(\frac{m}{t(1-\alpha)}\right)^{G_t}-1}{G_t}(1-\min(0,G_t))\geq 0\ when\ \frac{m}{t(1-\alpha)}\geq 1$,

Using (25), (26), (27) and (28), we get if $K\to+\infty$

$$VaR_\alpha=\left\{\frac{\omega}{1-\alpha_1-\beta_1}\right\}^{0.5}\left\{Z_{(T-m)}+D\left(\frac{m}{T(1-\alpha)}\right)C_m(T)\right\}$$

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