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HEDGING STRATEGY COMPARISONS OF VOLATILITY INDEX OPTIONS USING DIFFUSION MODELS

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ABSTRACT

With the innovation of derivatives, the Standard and Poor's (S&P) 500 index -- as an underlying asset of the volatility index (VIX) introduced by the Chicago Board Options Exchange (CBOE) -- was adopted as the research subject in this study. Since the financial crisis of 2008, the degree of market volatility has increased substantially. In addition, a random process has been found jumping about in the VIX data. In this study we compare VIX options based on different diffusion models. In this study, when a jump component is considered in the VIX process, the expectation maximization (EM) method is used to estimate parameters; this is a different perspective of evaluation from other studies. This paper further analyzes different hedging strategies based on different diffusion models.

JEL: G13, G17

KEYWORDS: VIX, Jump Process, MLE, EM Algorithm, Hedging Strategy

INTRODUCTION

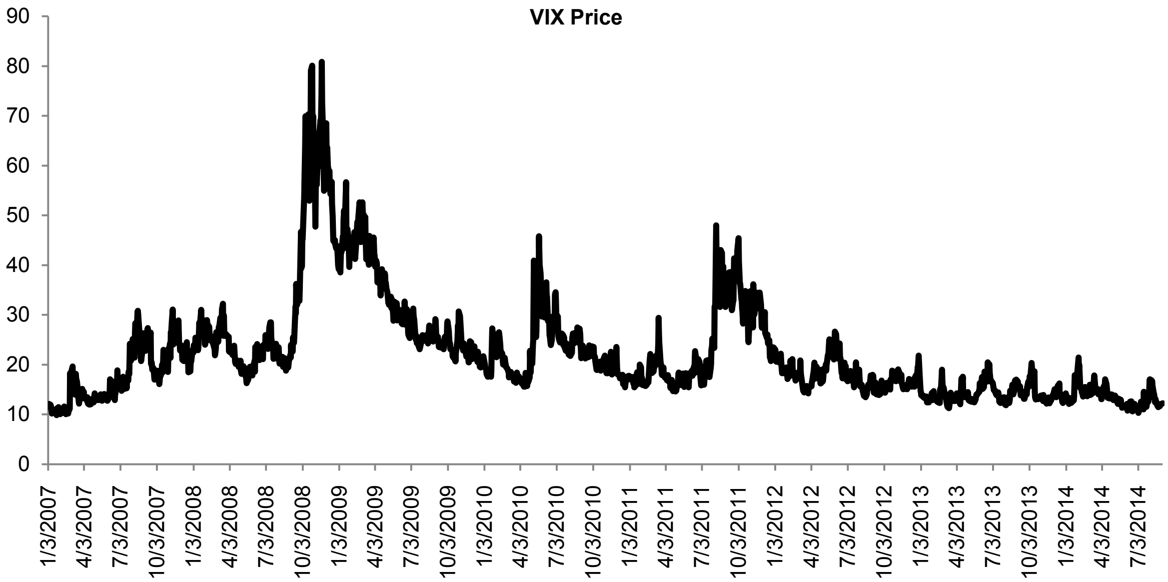
In recent years, volatility indices have been popular as a measure of market uncertainty. The first volatility index, VIX, was introduced by the Chicago Board Options Exchange (CBOE) in 1993. The VIX is calculated from Standard and Poor's (S&P) 500 option prices. In 2003, the CBOE modified the VIX by using the model-free methodology as a weighted sum of the out-of-the-money S&P 500 call and put option prices at two nearby maturities across all available strikes. In order to hedge volatility risk, Brenner and Galai (1989, 1993) first suggest the volatility derivatives. Until now, there are several types of derivatives used for hedging (or trading) volatility, including variance and volatility swaps, futures, and options. In this paper, we compare different hedging strategies of VIX options using different models. Specifically, unlike prior research, this study uses both maximum likelihood estimation (MLE) and expectation maximization (EM) algorithms to estimate parameters of diffusion processes. These two approaches are taken because recent studies have found that the MLE algorithm has some weakness in parameter estimations when the stochastic processes feature jump components.

We find that VIX option values, based on a diffusion model, will undervalue in a short time to maturity but overvalue in a long time to maturity. In addition, when investors consider hedging strategies, VIX options based on different diffusion models might influence the performance of the hedging strategies. The remainder of this paper is organized into five sections. Section 2 reviews the previous findings in the literature. In Section 3, we describe the methodology -- the EM algorithm. In Section 4, we present the numerical results for comparisons. A conclusion is provided in Section 5.

LITERATURE REVIEW

In the related literature, various volatility option-pricing models have been developed. For example, Whaley (1993), and Detemple and Osakwe (2000) use different specifications of the process that the VIX may follow. Bollerslev, Kretschmer, Pigorsch and Tauchen (2009), as well as Aboura and Wagner (2014) use the ARCH model for volatility of the volatility of daily market returns. Kaeck and Alexander (2012) estimate several continuous-time models by using the Markov chain Monte Carlo and then tested parameter estimates with extensive option data samples. Hao and Zhang (2013) propose a joint likelihood estimation with returns and VIX for option pricing. There has also been a growing interest in the literature concerning modeling the time series dynamics of implied volatility processes. For example, Daouk and Guo (2004) estimate mean-reverting processes from implied volatility indices. Bakshi *et al*. (1997) estimate various diffusion processes with a non-linear drift and a diffusion component on the square of VXO. Although previous studies focus on the dynamic process of implied volatility, most of them follow a standard Wiener process. However, it is obvious that the VIX process might elicit a jump component from the empirical data (see Figure 1).

Figure 1: Time Series Process of VIX



Note: This figure depicts the daily VIX price from 1/2007 to 8/2014

Due to this phenomenon, Wagner and Szimayer (2004) estimate a mean reverting jump diffusion process using the VIX and VDAX (VDAX is the implied volatility of German stock index) approaches. They find that positive jumps exist in implied volatilities. However, in their study, a constant jump size is used, which might affect the significance of jumps. Dotsis *et al*. (2007) examine the ability of alternative popular continuous-time diffusion and the jump diffusion processes to capture the dynamics of volatility indices. They find that the best fit to the data was the model featuring random jumps. Sepp (2008) model the VIX with the dynamics of the variance of the S&P 500 and find that jumps are important in variance. Psychoyios, Dotsis, and Markellos (2009) argue that a mean reverting logarithmic diffusion with jumps could successfully capture the VIX process. Wang and Daigler (2011) compare the empirical fit of the square root process (SQR) and Geometric Brownian Motion (GBM) models using data on options written on the VIX process. They also find evidence supporting the GBM assumption.

In related literature, researchers have addressed the jump component in other underlying asset processes. Zhou (2001) take the jump risk into account in his model and find that this jump-diffusion model is consistent with the fact that bond prices often drop at or around the time of default. Although a jump process is included in Zhou's model, there is still another problem that needed to be solved. In his model, the unobservable market value of a firm's assets is created by a Monte Carlo approach; this asset value is used to value a risky discount bond. How to get a more precise asset value becomes an important issue. Duan (1994) provide the MLE for evaluating asset values. By transferring the observable market equity data from the theoretical equity pricing formula, we can obtain the unobservable asset values. This transformed-data MLE method has been applied in Brockman and Turtle (2003), Duan, Gauthier, and Simonato (2003), Duan, Gauthier, and Simonato (2004), Ericsson and Reneby (2005), and Wong and Choi (2009).

However, this MLE method has failed when jumps appear in the diffusion process. Craine, Lochstoer, and Syrtviet (2000) point out that when discontinuous jumps exist, the MLE is not appropriate often because local maxima are present and the algorithm would not always converge. Wong and Li (2006) argue that the MLE fails when jumps exist in credit risk models. They mention that two problems may arise when the MLE method is used for jump-diffusion structural models. First, the optimization algorithm keeps running for a very long time without converging to a stable solution. This occurrence is due to the act that the global maximum is indeed infinite. Second, an unreasonably small volatility is obtained because the volatility nears zero when the program is run for a long time.

Recent studies have shown that when occasional discontinuous jumps occur, the MLE algorithm does not always converge. The EM algorithm is then introduced. This EM algorithm is used by Duan, Gauthier, and Simonato (2004) and Wong and Li (2006) to deal with the problems that arise when the jump processes of asset values are adopted to price equity default swap (EDS). They argue that if jump components are considered, the EM algorithm is numerically more robust than the direct maximum likelihood.

DATA AND METHODOLOGY

The VIX daily data are obtained from the CBOE. A total of 2,265 observations are collected during the data period that ran from 2004 to 2012. In this study, we take the VIX data in order to estimate the parameters of diffusion processes; several diffusion processes are used to capture the VIX process. Let $\mathrm{V_t}$ serve as the value of the implied VIX at time t, and let $\mathrm{dW_t}$ be a standard Wiener process. In order to simplify the model, we will first assume that the default-free interest rate is constant over time. The first process is the Merton process:

$$\mathrm{dlnV_t} = \mu\, d\mathrm{t} + \sigma\, d\mathrm{W}_t \tag{1}$$

As we mentioned before, Figure 1 shows the possibility of jumps in the VIX; hence, we also consider processes augmented with jumps.

$$\frac{dV_t}{V_{t^-}} = \mu\, d\mathrm{t} + \sigma\, dW_t + d\left(\sum_{j=1}^{N_t}(Z_j - 1)\right) \tag{2}$$

By using the $It\hat{o}'s\ lemma$, we can have:

$$d\omega_t = \left(\mu - \frac{1}{2}\sigma^2\right) d\mathrm{t} + \mathrm{\sigma}\, d\mathrm{W}_t + Y dN_t \text{ , } \omega_t = \mathrm{log}V_t \tag{3}$$

$\mu = r - \lambda m,\ m = \mathrm{E}[Z] - 1$, Z is the jump size, which follows log-normal distribution, $Y = \ln Z \sim N(k, s^2)$. dN_t presents the arrival of unexpected events, following the Poisson process, which means λ. $d\mathrm{W}$, Y and $\mathrm{dN_t}$ are mutually independent. As previously mentioned, the MLE method has some disadvantages in estimating processes with jump components. In order to have a consistent analysis, we adopt the EM algorithm for both processes with and without jumps. Here we briefly introduce how the EM algorithm can be used for variable estimations under jump processes.

The EM algorithm is typically used to compute maximum likelihood estimates given incomplete data like a jump process or hidden variables. Let y be the observed data from a p.d.f. of $f(y, \varphi)$, where $\varphi =$

$(\varphi_1, \varphi_2, \dots, \varphi_d)$ is a vector of parameters. Let x = [y, z] be a vector of complete data with the augmented data z. The incomplete data vector y comes from the incomplete sample space $y \in Y$. There is a 1-1 correspondence between the complete sample space x and the incomplete sample space y.
Let $\varphi^{(0)}$ be some initial value for φ. At the k-th step, the EM algorithm performs the following two steps:
E-step: Projecting an appropriate functional containing the complete data on the space of the incomplete data. Calculate:

$$Q\left(\varphi, \varphi^{(k)}\right) = E_{\varphi^{(k)}}(logL(\varphi|x)|y)$$

M-step: Maximizing the functional evaluated in the E-step.

Choose the value $\varphi^{(K+1)}$ that maximizes:

$$Q\left(\varphi, \varphi^{(k)}\right), ie., Q\left(\varphi^{(k+1)}, \varphi^{(k)}\right) \geqq Q\left(\varphi, \varphi^{(k)}\right)$$

The E and M steps are iterated until the difference of $\mathrm{L}\left(\varphi^{(K+1)}\right) - L(\varphi^{(K)})$ becomes small enough. Based on the above idea, we need the conditional p.d.f. of ω_t which is:

$$g(\omega_i|\omega_{i-1}) = (1 - \lambda\Delta t_i) f_X(\omega_i|\omega_{i-1}) + \lambda\Delta t_i f_{X+Y}(\omega_i|\omega_{i-1}) \quad (4)$$

Where:

$$X|_{\omega_{i-1}} \sim N(\omega_{i-1} + \tilde{\mu}, \sigma^2\Delta t_i) \text{ , } \tilde{\mu} = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t_i$$

$$(X + Y)|_{\omega_{i-1}} \sim N(\omega_{i-1} + \tilde{\mu} + k, \sigma^2\Delta t_i + s^2)$$

Here, we define$\mathrm{C} = \{c_n \in \{0,1\}, n = 1, \dots, N\}$. $c_n = j$represents$j$ times jumps in the (t_{n-1}, t_n) interval. Let $\pi_0 = (1 - \lambda\Delta_t)$ be the probability that no jump happens in the time interval Δ_t, so $\pi_1 = 1 - \pi_0$ indicates the probability for the jump. Wong and Li (2006) derived the EM algorithm by using the re-estimation formula. Following their steps, the re-estimation formula for all variables include (Please refer to Wong and Li (2006) for more details.):

$$\pi_0^{(m+1)} = \frac{1}{N}\sum_{j=1}^{N} P\left(c_j = 0\middle|V^{(m)}, \theta^{(m)}\right) \quad (5)$$

Where:

$$P\left(c_j = 0\middle|V^{(m)}, \theta^{(m)}\right) = \frac{\pi_0^{(m)} f_X\left(\omega_j^{(m)}\middle|\omega_{j-1}^{(m)}, \mu^{(m)}, \sigma^{(m)}\right)}{\left(\omega_j^{(m)}\middle|\omega_{j-1}^{(m)}, \theta^{(m)}\right)} \quad (6)$$

RESULTS AND DISCUSSION

Numerical Results

In order to examine whether our program can correctly estimate the needed parameters, we first create time series data based on some given parameters and then use the EM algorithm to check the accuracy. We repeat the estimation 100 times and the results are shown in Tables 1 and 2. We simulate a time series process based on the stochastic process we mentioned. For the Merton diffusion process, we can rewrite it as:

$$V_t = V_{t-1} exp\left[(\mu - 0.5\sigma^2)dt + \sigma\varepsilon\sqrt{dt}\right]$$

For the Merton jump diffusion process, we have:

$$V_t = V_{t-1} exp\left[(\mu - \lambda m - 0.5\sigma^2)dt + \sigma\varepsilon\sqrt{dt}\prod_{j=1}^{N} Z_j\right]$$

Table 1: The Comparisons between MLE and EM

	MLE_NO JUMP	EM_NOJUMP	EM_JUMP
μ=0.08	0.0730	0.0744	0.0777
σ=0.3	0.2994	0.2994	0.3025
λ=10			10.280
k=-0.05			-0.0556
s=0.1			0.0972
Likelihood function	664.63	664.63	675.97
AIC	-1325.3	-1325.3	-1341.9
BIC	-1310.8	-1310.4	-1322.4

Note: This table shows the comparisons between MLE and EM. The parameters are as follows: $\mu = 0.08, \sigma = 0.3, \lambda = 10, \kappa = -0.05,\ s = 0.1$

Table 1 shows the simulation results. From the above results, it seems that the EM algorithm has good parameter estimation results. Even though we use the EM algorithm for parameters estimation without a jump component process, we still have a lower AIC and BIC. Since the EM algorithm can provide better results for parameters estimation, we then use the EM algorithm to estimate parameters for diffusion models by using the real VIX data. The results are shown in Table 2.

Table 2: The Parameters Estimation of the VIX Process

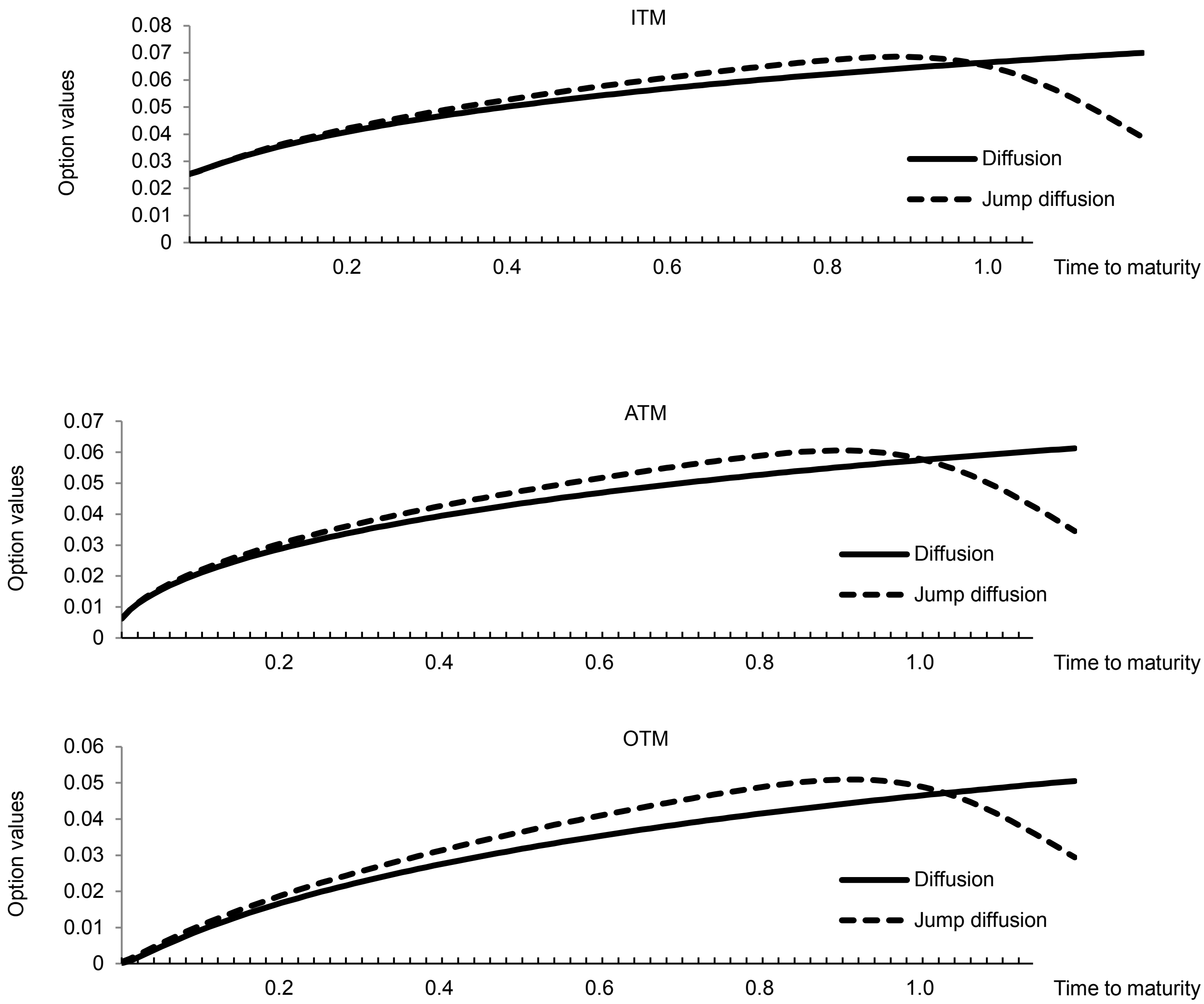
	EM_NOJUMP	EM_JUMP
μ	0.5686	-0.8115
σ	1.0676	0.7612
λ		39.405
k		0.0375
s		0.1193

Note: Table 2 shows the parameters estimation of VIX based on the EM algorithm.

The Relationship of Call Options and Time to Maturity

We can use the parameters obtained above for options pricing to demonstrate the performance of hedging strategies based on different diffusion process assumptions. We show the comparisons of in-the-money (ITM), out-the-money (OTM), and at-the-money (ATM) options based on different diffusion process settings. In Figure 2, we assume the diffusion model and jump-diffusion model for the VIX to gain the relation of time to maturity along with 20 percent ITM call options, 20 percent OTM, or 20 percent ATM call options. We find that when the time is getting closer to expiration, the price of 20 percent ITM call options are highest, followed by ATM call options. Furthermore, when compared with the jump-diffusion model, the diffusion models of 20 percent ITM, 20 percent OTM, and 20 percent ATM call options are undervalued with shorter maturities but overvalued with longer maturities.

Figure 2: The Relation between Option Values and Time to Maturity

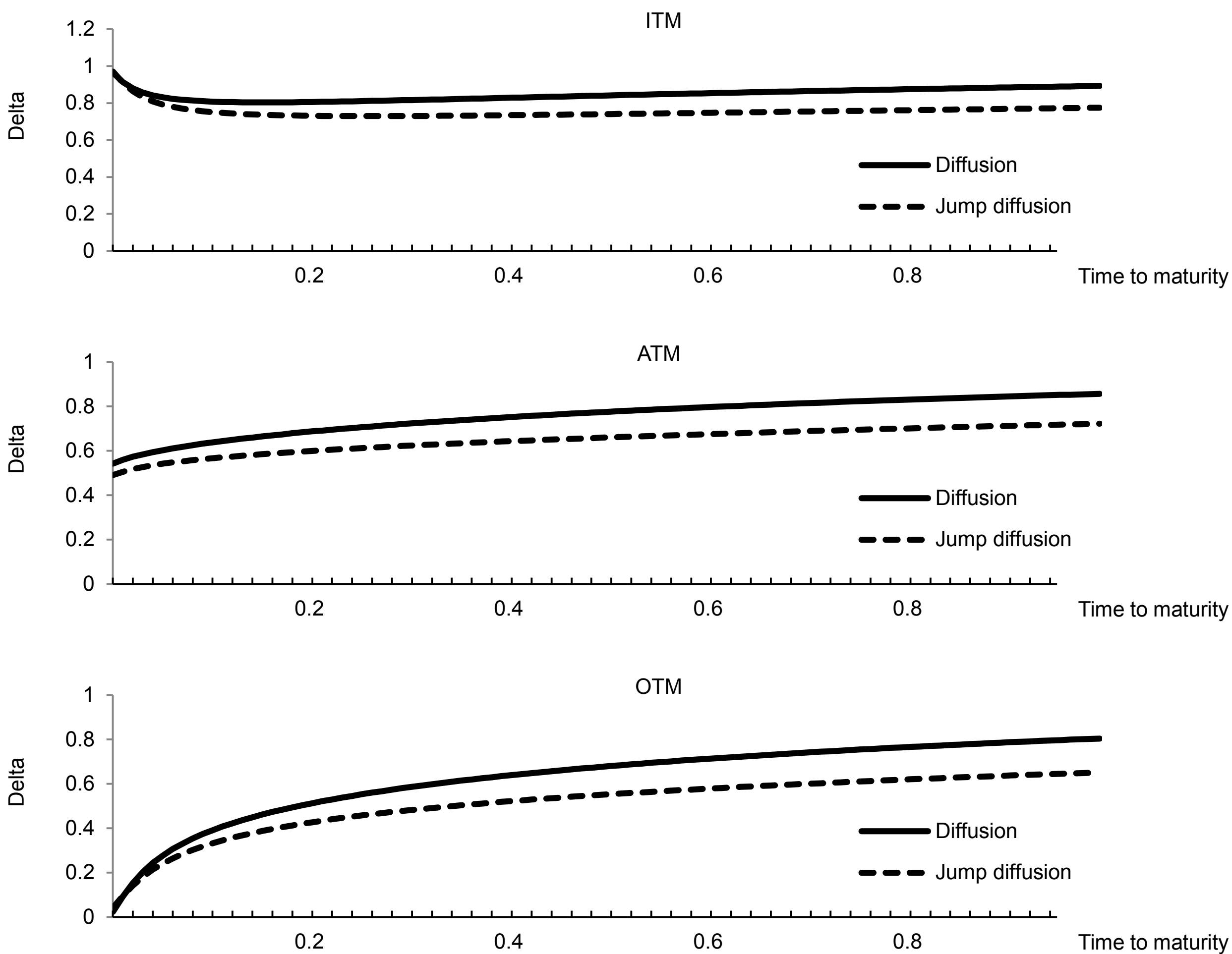


Note: This figure shows the relation between the Delta of VIX options and time to maturity of a 20% ITM call option, a 20% OTM, and a 20% ATM call option based on a diffusion model and jump-diffusion model to price VIX options. We assume r=5% and V_t*=15%.*

Sensitivity Analysis

We also compare different hedge strategies of call options based on different diffusion models. In Figure 3, we use a diffusion model and jump-diffusion model to estimate the Delta of VIX options. We then obtain the relation of time to maturity and 20 percent ITM call options, 20 percent OTM call options, or 20 percent ATM call options. Figure 3 shows that diffusion models are all higher than jump-diffusion models on any conditions as times goes by. In addition, we find that the longer the time to maturity, the bigger the difference between diffusion and jump diffusion process tends to be.

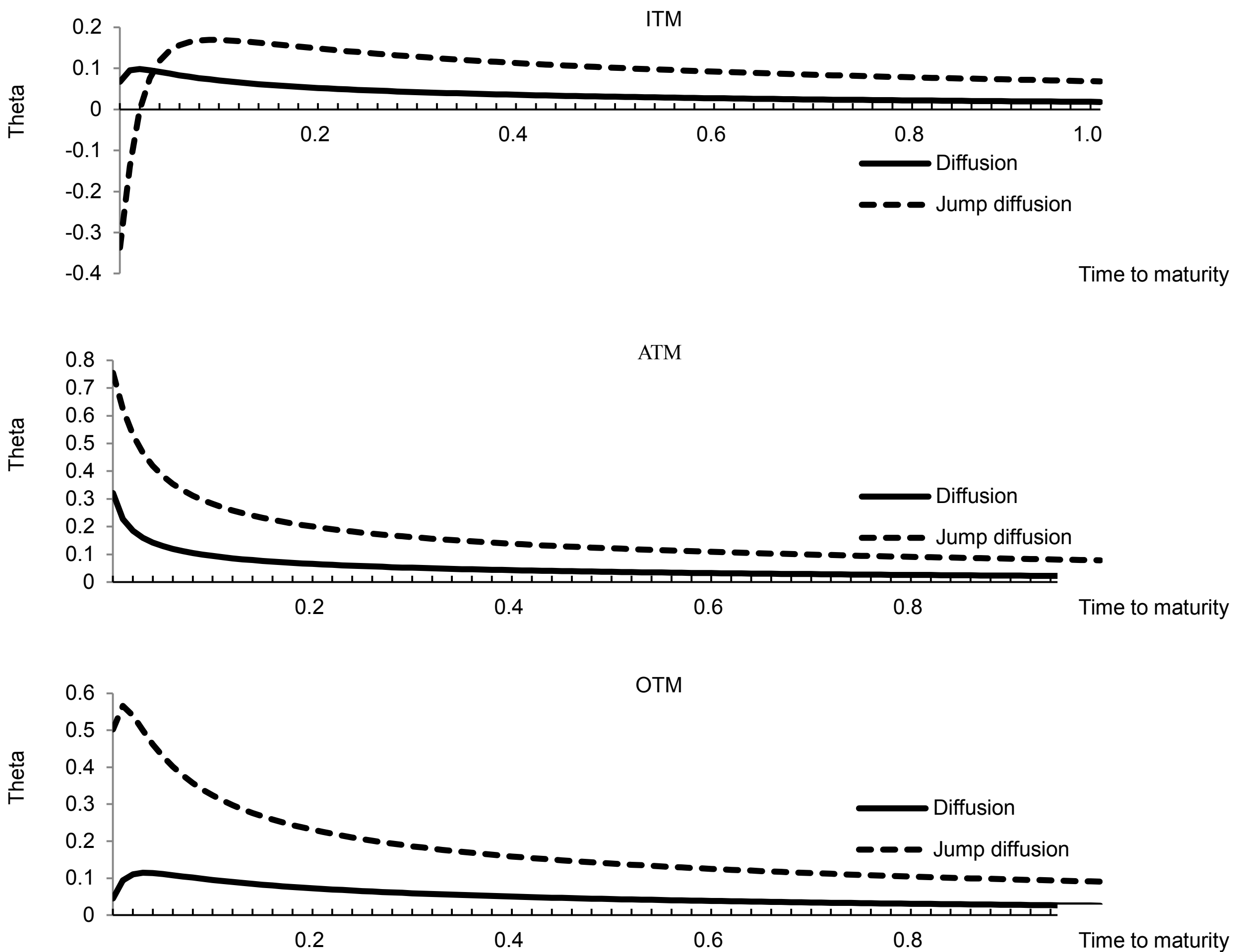
Figure 3: The Relation between Delta and Time to Maturity



Note: This figure shows the relation between the Delta of VIX options and time to maturity of a 20% ITM call option, a 20% OTM, and a 20% ATM call option based on a diffusion model and a jump-diffusion model to price VIX options. We assume r=5% and V_t*=15%.*

In Figure 4, we use the diffusion model and jump-diffusion model and estimate the Theta of VIX options to gain the relationship of time to maturity and 20 percent ITM call options, 20 percent OTM call options, or 20 percent ATM call options. The picture shows that diffusion models manifest higher than jump-diffusion models as times goes on, but lower than jump-diffusion models when meeting a specific time point. However, diffusion models are not more significant than jump-diffusion models if the call is 20 percent ATM and 20 percent ITM.

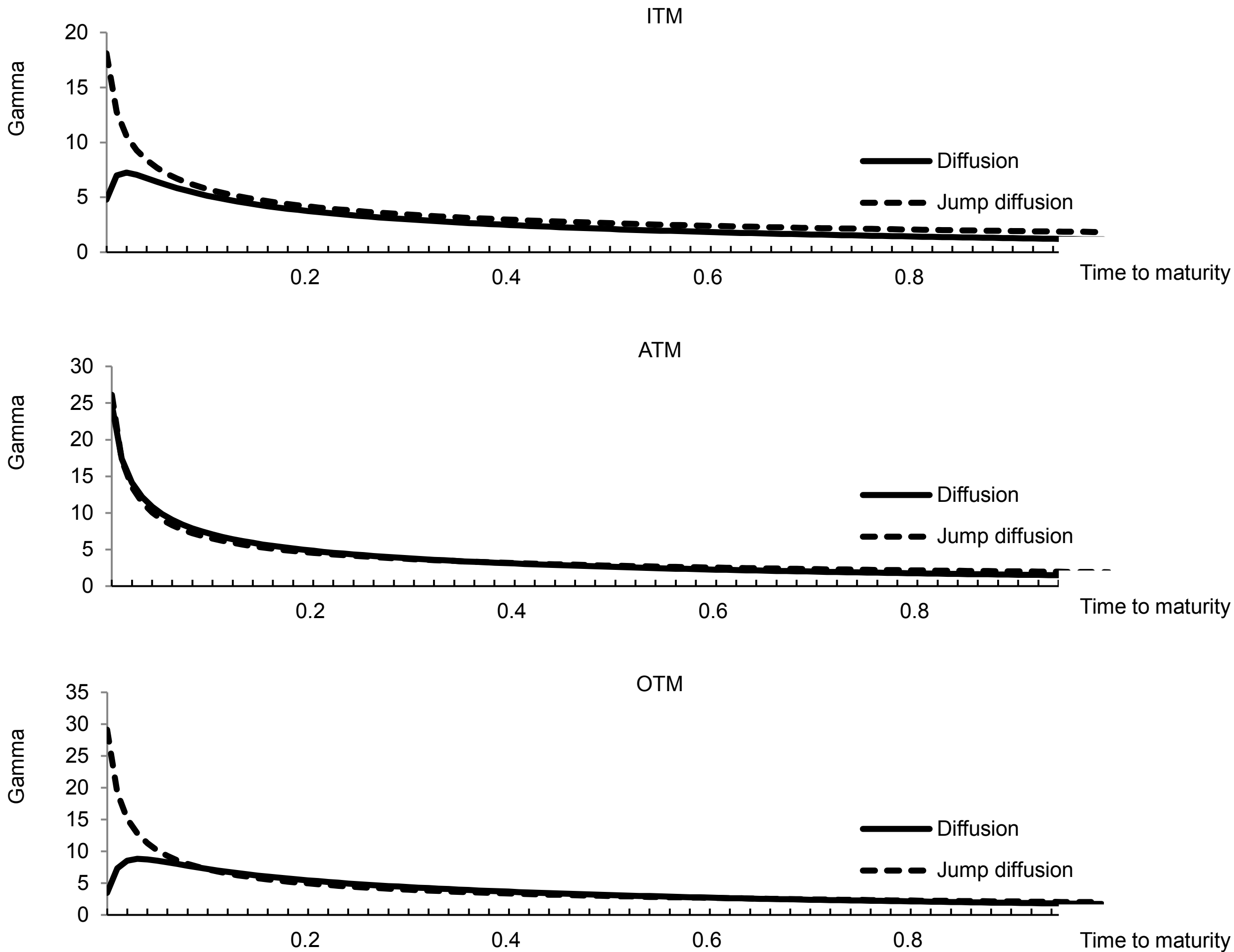
Figure 4: The Relation between Theta of VIX Options and Time to Maturity



Note: This figure shows the relation between the Theta of VIX options and time to maturity of a 20% ITM call option, a 20% OTM call option, and a 20% ATM call option based on a diffusion model and a jump-diffusion model to price VIX options. We assume r=5% and V_t*=15%.*

Figure 5 shows the relation between the Gamma of VIX options and time to maturity of ITM, OTM, and ATM call options. We find that diffusion models are apparently lower than jump-diffusion models as times goes by. In addition, options based on these two different diffusion models are getting close as it takes longer to reach maturity.

Figure 5: The Relation between Gamma of VIX Options and Time to Maturity



Note: This figure shows the relation between the Gamma of VIX options and time to maturity of a 20% ITM call option, a 20% OTM call option, and a 20% ATM call option based on a diffusion model and a jump-diffusion model to price VIX options. We assume r=5% and V_t*=15%.*

CONCLUSION

In this paper, we diverge from previous studies and compare VIX options based on a diffusion model (Merton 1974) and a jump-diffusion model (Merton 1976). The reason we consider a jump component is that there are occasional jumps in the VIX process. By comparing AIC and BIC, we find that using an EM algorithm for a jump-diffusion model is more suitable than for a diffusion model. Since VIX options are usually used for hedging strategies, and to have a more general comparison, we show the relationship between VIX option Greeks based on different diffusion models and time to maturities. The results show that VIX option values based on a diffusion model will undervalue in a short time to maturity, but they will overvalue over a long duration to maturity. In addition, hedge ratios also show various levels of differences based on a diffusion model and a jump-diffusion model. This implies that when investors consider hedging strategies, VIX options based on different diffusion models might influence the strategies' performances. A worthy issue for future research would be to consider more complicated diffusion models for comparison such as a mean reverting model and a stochastic volatility model.

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